ERROR BOUND FOR BERNSTEIN-BÉZIER TRIANGULAR APPROXIMATION*

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Abstract

Based upon a new error bound for the linear interpolant to a function defined on a triangle and having continuous partial derivatives of second order, the related error bound for n-th Bernstein tirangular approximation is obtained. The order of approximation is $1/n$.

1. Introduction

Bernstein-Bézier polynomials, or surfaces, have been studied extensively[1-5]. In this paper we first present an error bound on the right-hand side of (12) and show that the coefficient 1 is the best. Then, based on (12), the error bound for the Bernstein–Bézier triangular approximation is obtained and the coefficient 1 is again proved to be the best.

2. Definition and Notation

We begin with a brief discussion on the area coordinates of points with respect to a given triangle. Let T be a triangle with vertices $T_\alpha=(x_\alpha, y_\alpha)$, $\alpha=1, 2, 3$, and area $|\Delta|$. An internal point $P=(x, y)$ of T divides the triangle $T_1T_2T_3$ into three smaller ones, PT_2T_3, PT_3T_1, PT_1T_2, with respective areas $|\Delta_1|$, $|\Delta_2|$, $|\Delta_3|$, which may vary from zero to $|\Delta|$, depending on the position of P. In other words, the ratios $u:=\frac{|\Delta_1|}{|\Delta|}$, $v:=\frac{|\Delta_2|}{|\Delta|}$, $w:=\frac{|\Delta_3|}{|\Delta|}$ will take up any value between zero and unity. Here (u, v, w) with $u+v+w=1$ are called area coordinates of the point P.

It is easy to see that

$$\begin{bmatrix}1\\x\\y\end{bmatrix}=\begin{bmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}. \tag{1}$$

Let $F(x, y)$ be a function defined on T, where x and y are Cartesian coordinates; the related function f dependent on area coordinates u, v, w is given by

$$f(u, v, w)=F(x_1u+x_2v+x_3w,\ y_1u+y_2v+y_3w). \tag{2}$$

The n-th Bernstein–Bézier polynomial of the function f over the triangle T is given by

$$B_n(f; u, v, w)=\sum_{i+j+k=n} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)J^n_{i,j,k}(u, v, w), \tag{3}$$

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where

$$J^n_{i,j,k}(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k \tag{4}$$

and i, j, k designate nonnegative integers such that $i+j+k=n$. Functions in (4) are called Bernstein basis polynomials of degree n, for they form a basis for all bivariate polynomials of degree n. In some cases $B_n(f; u, v, w)$ and $f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)$ are replaced by $B_n(f)$ and $f_{i,j,k}$ for simplicity.

We outline the basic properties of $B_n(f)$ as follows:

(a) B_n is a positive and linear operator carrying every function defined on T to a bivariate polynomial of degree n.

(b) $B_n(f)$ interpolates to f at the three vertices of T, i. e. $B_n(f; T_\alpha) = f(T_\alpha)$, $\alpha = 1, 2, 3$.

(c) Since the functions $J^n_{i,j,k}$ in (4) are nonnegative on T and sum to $(u+v+w)^n = 1$, each point on the Bernstein–Bézier triangular surface is a convex combination of $f_{i,j,k}$. Hence we can say that the surface (3) lies within the convex hull of the points $P_{i,j,k}: = \left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}; f_{i,j,k}\right)$, $i+j+k=n$, which are on the surface associated with the function f.

(d) If f is a continuous function on T, then

$$\lim_{n\to\infty} B_n(f; u, v, w) = f(u, v, w) \tag{5}$$

uniformly on T (see, e. g., p. 51 of [1]).

(e) Simple calculation shows

$$J^n_{i,j,k} = \frac{1}{n+1}[(i+1)J^{n+1}_{i+1,j,k} + (j+1)J^{n+1}_{i,j+1,k} + (k+1)J^{n+1}_{i,j,k+1}], \tag{6}$$

which enables us to express $B_n(f)$ in terms of the Bernstein basis polynomials of degree $n+1$:

$$B_n(f) = \frac{1}{n+1} \sum_{i+j+k=n+1} (if_{i-1,j,k} + jf_{i,j-1,k} + kf_{i,j,k-1}) J^{n+1}_{i,j,k}. \tag{7}$$

Concerning the surface points $P_{i,j,k}$, $i+j+k=n$, in (c), we further note that there are altogether $\frac{(n+1)(n+2)}{2}$ such points in the space. If a line segment is connected each two of the three points

$$P_{i+1,j,k}, \quad P_{i,j+1,k}, \quad P_{i,j,k+1},$$

where $i+j+k=n-1$, a piecewise linear function on T is obtained, which is denoted by $\hat{f}_n(u, v, w)$. $\hat{f}_n$ is called the n-th Bézier net of f, in accordence with literature in CAGD (see Farin[3]).

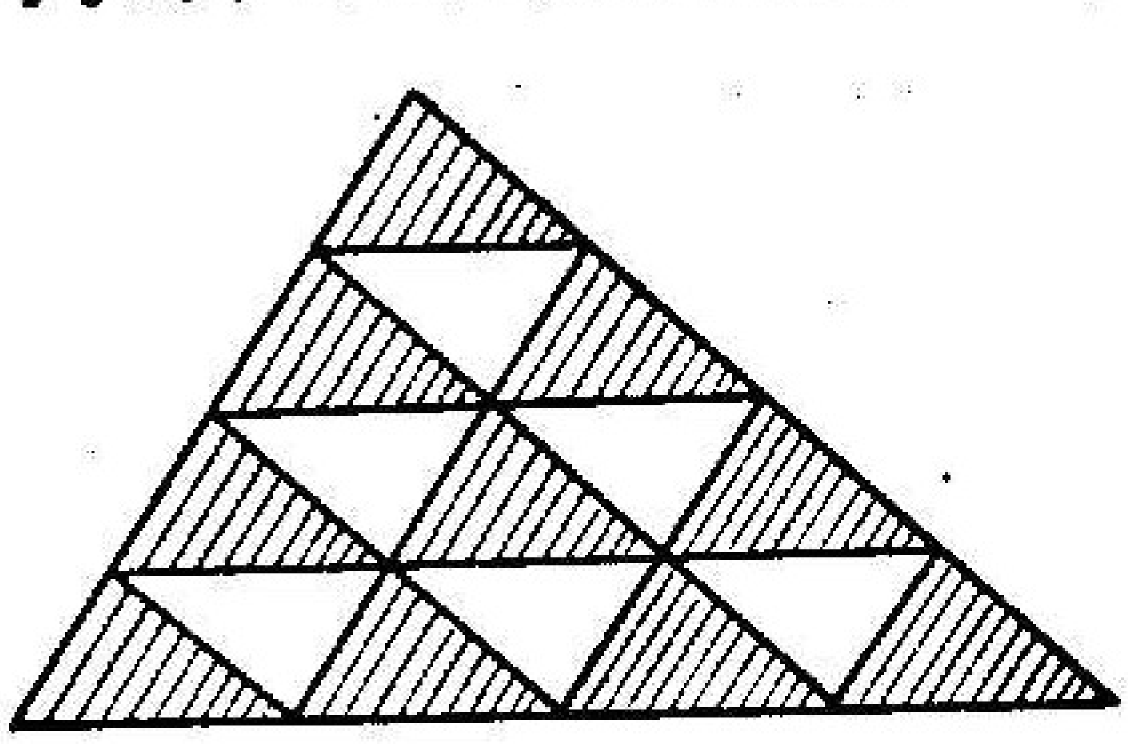

Fig. 1

The projection of $\hat{f}_n$ onto the triangle T produces a subdivision of T, denoted by $S_n(T)$. Each of the points $\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)$ satisfying $i+j+k=n$ is called a node of $S_n(T)$. $S_4(T)$ is illustrated in Fig. 1.

The subtriangles in $S_n(T)$ can be divided into two categories: