THE HIGH ORDER EXPONENTIALLY FITTED NONEQUIDISTANT EXTRAPOLATION METHODS FOR STIFF SYSTEMS*

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Abstract

A class of exponentially fitted nonequidistant extrapolation methods based on an L-stabe linear single-step formula are studied. Theorems about the extrapolation coefficients are given. These methods keep good numerical stability, "quasi-L-stability", while raising the accuracy of the original formula.

§ 1. Introduction

In solving the initial value problem of O. D. E. s

$$\begin{cases} y'=f(y), \\ y(a)=\eta \end{cases}$$

the stiffness will become a serious difficulty when the ratio of the real part of the eigenvalues of the Jacobian $J=\frac{\partial f}{\partial y}$ is very large. Because of the importancy of this class of problems, the research for efficient computing methods is significant.

The technique of extrapolation has been widely used, but most such methods destroy the numerical stability when they raise the accuracy, and the amount of work increases as exponential of 2. J. R. Cash[4] introduced exponential fitted extrapolation methods to improve the stability of local extrapolation, but the eigenvalues of the Jacobian must be calculated, and this costs a lot of machine time.

In this paper, a class of exponentially fitted nonequidistant extrapolation (EFNE) methods are discussed, and the calculation of eigenvalues is no longer needed. With the use of the strategy of nonequidistant extrapolation, the amount of work increases linearly. Another advantage is that the new methods have quasi-L-stability, which is analogous to L-stability.

§ 2. The Construction of EFNE Methods

Consider the linear single-step formula of order 3:

$$y_{n+1}=y_n+\frac{1}{3}h(2f_{n+1}+f_n)-\frac{1}{6}h^2f'_{n+1}. \tag{1}$$

When it is applied to the test equation

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$$y'=\lambda y, \quad y(0)=1 \tag{2}$$

the characteristic function is

$$R_2^1(q)=\frac{1+q/3}{1-2q/3+q^2/6}, \tag{3}$$

where $q=\lambda h$. It is well known that (3) is L-acceptable and therefore (1) is L-stable. Now we try to construct new methods of higher accuracy based on (1). Assume that $y_1, \cdots, y_n$ have been calculated and the $(n+1)$th main step is to compute y_{n+1}. Let h be the step-length of this step, choose real numbers $m_i\geq 1$ $(i=1, \cdots, m)$, and calculate y_{n+1} for m times (each calculation denoted as $y_{n+1}^{(i)}$):

$$\begin{aligned} y_{n+\frac{1}{m_i}}&=y_n+\frac{1}{3}\frac{h}{m_i}(2f_{n+\frac{1}{m_i}}+f_n)-\frac{1}{6}\left(\frac{h}{m_i}\right)^2 f'_{n+\frac{1}{m_i}}, \\ y_{n+1}^{(i)}&=y_{n+\frac{1}{m_i}}+\frac{1}{3}\left(\frac{m_i-1}{m_i}h\right)(2f_{n+1}^{(i)}+f_{n+\frac{1}{m_i}})-\frac{1}{6}\left(\frac{m_i-1}{m_i}h\right)^2 f_{n+1}^{(i)\prime}. \end{aligned} \tag{4}$$

Generally, let $m_1=1$, $m_i\neq m_j$, $i\neq j$. Then take the linear combination

$$y_{n+1}=\sum_{i=1}^{m} u_i y_{n+1}^{(i)}. \tag{5}$$

Connect (4), (5) using (3), and we have the characteristic function of the EFNE methods:

$$R(q)=\sum_{i=1}^{m} u_i R_2^1\left(\frac{q}{m_i}\right) R_2^1\left(\frac{m_i-1}{m_i}q\right). \tag{6}$$

Choose u_i so that $R(q)$ can approximate to e^q as closely as possible, i.e.

$$R(q)=e^q+O(q^{p+1}), \tag{7}$$

where p (positive integer) should be as large as possible.

Definition 1. *An EFNE method is of order p if*

$$R(q)=e^q+c_{p+1}q^{p+1}+O(q^{p+2}), \tag{8}$$

where $c_{p+1}\neq 0$ is the error constant of the method.

The following theorems show how u_i's are chosen and how m is confined.

These theorems are generally true for exponentially fitted nonequidistant extrapolation methods based on formulae different from (1).

Theorem 2. *Let $m_i>1$, $i=2, \cdots, m$, and $m_i\neq m_j$, $i\neq j$. Denote*

$$a_{ij}=\frac{1+(m_j-1)^{i+2}}{m_j^{i+2}}, \quad i, j=2, \cdots, m\leq 4, \tag{9}$$

$$A=\begin{bmatrix} 1 & 1\cdots 1 \\ 1 & \\ \vdots & a_{ij} \\ 1 & \end{bmatrix}, \quad e_1=(1, 0, \cdots, 0)^T$$

Then the coefficients u_i in (6) can be determined by

$$Au=e_1. \tag{10}$$

Proof. Let

$$\begin{aligned} R_j&=R_2^1\left(\frac{q}{m_j}\right) R_2^1\left(\frac{m_j-1}{m_j}q\right) \\ &=\frac{(1+q/3m_j)(1+(m_j-1)q/3m_j)}{(1-2q/3m_j+(q/m_j)^2/6)(1-2(m_j-1)q/3m_j+((m_j-1)q/m_j)^2/6)}, \end{aligned}$$