APPROXIMATION OF BOUNDARY CONDITIONS AT INFINITY FOR A HARMONIC EQUATION*

YU DE-HAO (余德浩)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

Starting from the canonical boundary reduction, this paper studies an approximate differential boundary condition and an approximate integral boundary condition on an artificial boundary for the exterior problem of a harmonic equation, and gives an error estimate for the latter. This estimate reveals the relationship between the error and the approximate grade of boundary conditions as well as the radius of the artificial boundary.

§ 1. Approximation of the Integral Boundary Condition

The treatment of an elliptic boundary value problem over an unbounded domain by the classical finite element method is often a difficulty, because a simple replacement of the infinite domain by a bounded domain can hardly produce the demanded accuracy. The canonical boundary reduction suggested by Feng Kang[1,2] and the coupling of the canonical boundary element method with the finite element method[5] have provided an approach to this problem.

Consider the boundary value problem of a harmonic equation over an exterior domain Ω with smooth boundary Γ_i

$$\begin{cases} -\Delta u=0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=f, & \text{on } \Gamma_i, \\ u \text{ is bounded at infinity}, \end{cases} \tag{1}$$

where $f\in H^{-\frac{1}{2}}(\Gamma_i)$ satisfies the compatibility condition. We draw a circle Γ_R with radius R enclosing Γ_i. Ω is divided into Ω_i and Ω_e (Fig. 1). Then the canonical integral equation on Γ_R, obtained from the harmonic boundary value problem over the exterior domain Ω_e by canonical boundary reduction, is just the exact boundary condition on the artificial boundary Γ_R of the original boundary value problem, i.e. the problem (1) is equivalent to

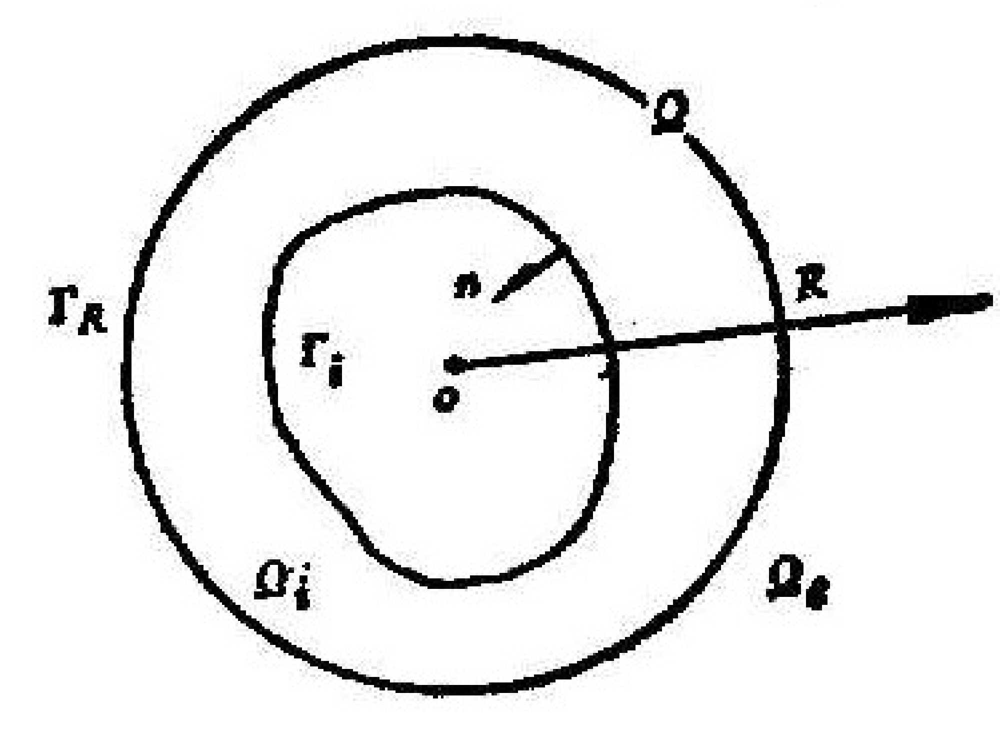

Fig. 1

* Received July 26, 1984.

$$\begin{cases} -\Delta u=0, & \text{in } \Omega_i, \\ \dfrac{\partial u}{\partial n}=f, & \text{on } \Gamma_i, \\ \dfrac{\partial u}{\partial r}(R,\theta)=\dfrac{1}{4\pi R\sin^2\dfrac{\theta}{2}} * u(R,\theta), & \text{on } \Gamma_R, \end{cases} \tag{2}$$

where Ω_i is the bounded domain between Γ_i and Γ_R, and $*$ denotes the convolution, which can be defined by the Fourier expansion. Because[7]

$$-\frac{1}{4\pi R\sin^2\dfrac{\theta}{2}}=\frac{1}{2\pi R}\sum_{-\infty}^{\infty}|n|e^{in\theta}=\frac{1}{\pi R}\sum_{n=1}^{\infty}n\cos n\theta,$$

the integral boundary condition of (2) can be written

$$\frac{\partial u}{\partial r}(R,\theta)=-\frac{1}{\pi R}\sum_{n=1}^{\infty}n\int_0^{2\pi}u(R,\theta')\cos n(\theta-\theta')\,d\theta'. \tag{3}$$

Obviously (3) is a non-local boundary condition, and its kernel is highly singular. We attempt to simplify this boundary condition for the sake of easier application. Using the asymptotic expansion method, [3] has obtained a series of asymptotic radiation conditions for reduced wave equations. These approximate boundary conditions are differential (i.e. local) boundary conditions. But this method is not applicable to the harmonic equation, so we naturally think of replacing (3) with an approximate integral boundary condition

$$\frac{\partial u}{\partial r}(R,\theta)=-\frac{1}{\pi R}\sum_{n=1}^{N}n\int_0^{2\pi}u(R,\theta')\cos n(\theta-\theta')\,d\theta', \tag{4}$$

where N is a positive integer. The integral kernel of (4) is nonsingular of course. In particular when the Fourier series of $u(R,\theta)$ only contains the first N terms, this boundary condition can be reduced into a local boundary condition

$$\frac{\partial u}{\partial r}(R,\theta)=\frac{1}{R}\sum_{k=1}^{N}\alpha_k\frac{\partial^{2k}}{\partial\theta^{2k}}u(R,\theta), \tag{5}$$

where $\alpha_k(k=1,\cdots,N)$ are the solution of

$$\sum_{k=1}^{N}(-n^2)^k\alpha_k=-n,\quad n=1,2,\cdots,N.$$

We call (4) and (5) an approximate integral boundary condition and an approximate differential boundary condition of grade N of (3) respectively. The first four approximate differential boundary conditions are as follows:

$$N=1:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\frac{\partial^2 u}{\partial\theta^2}, \tag{6$_1$}$$

$$N=2:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(\frac{7}{6}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{6}\frac{\partial^4 u}{\partial\theta^4}\right), \tag{6$_2$}$$

$$N=3:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(\frac{74}{60}\frac{\partial^2 u}{\partial\theta^2}+\frac{15}{60}\frac{\partial^4 u}{\partial\theta^4}+\frac{1}{60}\frac{\partial^6 u}{\partial\theta^6}\right), \tag{6$_3$}$$

$$N=4:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(\frac{533}{420}\frac{\partial^2 u}{\partial\theta^2}+\frac{43}{144}\frac{\partial^4 u}{\partial\theta^4}+\frac{11}{360}\frac{\partial^6 u}{\partial\theta^6}+\frac{1}{1008}\frac{\partial^8 u}{\partial\theta^8}\right). \tag{6$_4$}$$

Let $\quad D_I(u,v)=\int_{\Omega_i}\nabla u\cdot\nabla v\,dx,$