APPROXIMATION OF INFINITE BOUNDARY CONDITION AND ITS APPLICATION TO FINITE ELEMENT METHODS*

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Abstract

The exterior boundary value problems of Laplace equation and linear elastic equations are considered. A series of approximate infinite boundary conditions are given. Then the original problem is reduced to a boundary value problem on a bounded domain. The finite element approximation of this problem and its error estimate are obtained. Finally, a numerical example shows that this method is very effective.

§ 1. Introduction

Many boundary value problems of partial differential equations involving the unbounded domain arise in practical applications, such as coupling of structures with foundation and environment and fluid flow around obstacles. In finding the numerical solutions of this kind of problems, it is often a difficulty using the classical finite element method or finite difference method. In engineering, the usual method is to cut off the unbounded part of the domain and to set up an artificial boundary condition at the new boundary of the remaining bounded domain. For example, the Dirichlet condition and Neumann condition are often used for elliptic partial differential equations. In general, the artificial boundary condition at the new boundary is only a rough approximation of the exact boundary condition. Hence the remaining bounded domain must be quite large when high accuracy is required. It is still difficult to compute the numerical solution on a quite large domain.

Combining the finite element method and the classical analytical method, Han and Ying[1] proposed the local finite element method for solving the elliptic boundary value problem on an unbounded domain. An exterior boundary value problem of model equation $\Delta u=0$ has been considered. By cutting off the exterior domain of a circle and getting the exact boundary condition at the new boundary of the remaining bounded domain by the classical analytical method, the original problem is reduced to an equivalent boundary value problem on a bounded domain with integral boundary condition. This method is closely related to the method of coupling of F. E. M. and canonical boundary reduction proposed by Feng Kang[2,3]. Their difference is in the form of the canonical integral equations. But in both methods, the integrals have singular kernels, and thus they are not

* Received September 15, 1984.

readily available for compution. In this paper, exterior boundary value problems of Laplace equation and the linear elastic equations are considered. A series of approximate infinite boundary conditions are given and applied to the finite element method. The error estimate of the finite element approximate solution is obtained and a numerical example shows the effectiveness of this method.

§ 2. An Exterior Boundary Value Problem of Laplace Equation

2.1. The continuous problem

Let Γ_i be a bounded, simple closed curve in $\mathbb{R}^2$, and Ω be the unbounded domain with boundary Γ_i. Consider the following problem:

$$\begin{cases} -\Delta u=0, \ \Omega, \\ u|_{\Gamma_i}=f_i, \\ u \text{ is bounded, when } r\to+\infty. \end{cases} \tag{2.1}$$

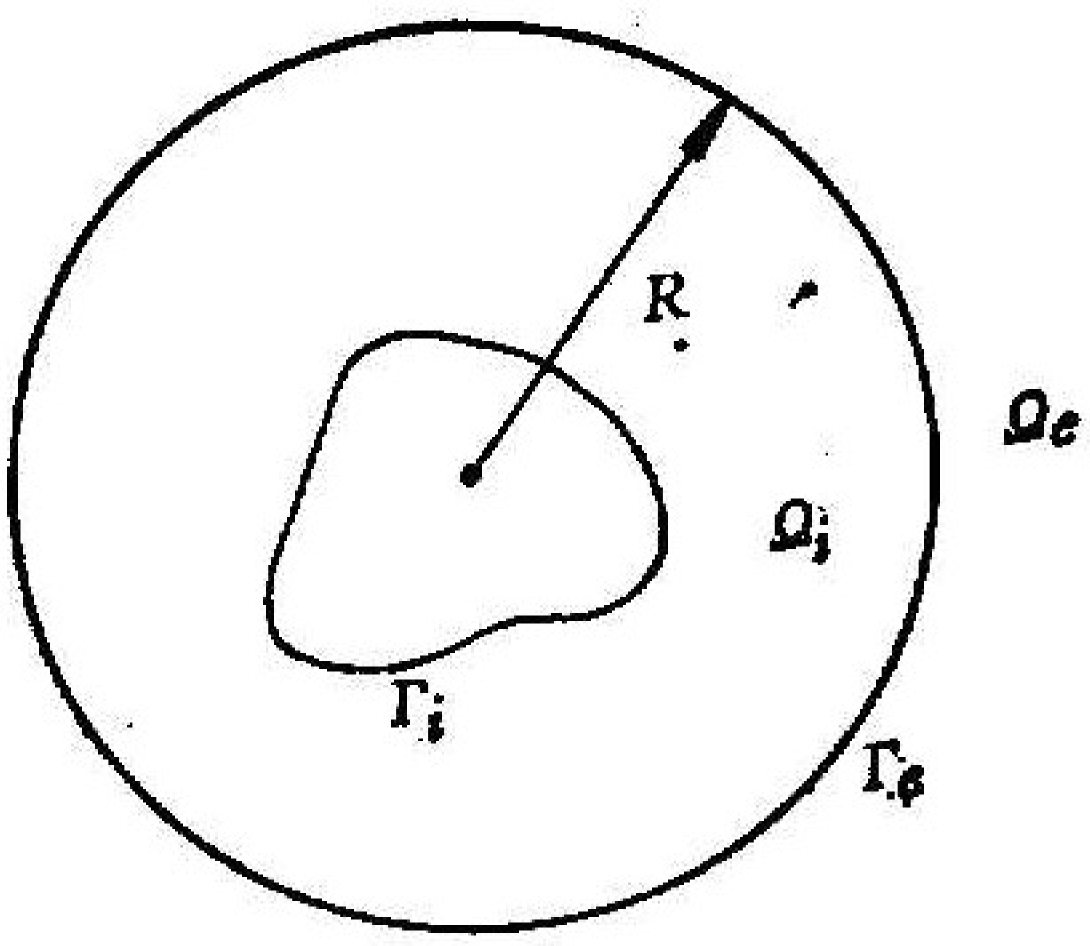

Fig. 1

This problem is defined on an unbounded domain Ω. First the problem is reduced to a boundary value problem on a bounded domain. In Ω, we draw a circumference Γ_e with radius R; then Ω is divided into two parts. The bounded part is denoted by Ω_i and $\Omega_e=\Omega\backslash\Omega_i$ is the unbounded part (see Fig. 1). Let $u(r, \theta)$ denote the solution of problem (2.1), where $x_1=r\cos\theta$, $x_2=r\sin\theta$. If a certain boundary condition of $u(r, \theta)$ on Γ_e is given, then we can consider problem (2.1) only on the bounded domain Ω_i. On domain Ω_e, $u(r, \theta)$ can be written as

$$u(r, \theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n(a_n\cos n\theta+b_n\sin n\theta), \tag{2.2}$$

therefore

$$\frac{\partial u(R, \theta)}{\partial r}=\sum_{n=1}^{\infty}-\frac{n}{R}(a_n\cos n\theta+b_n\sin n\theta). \tag{2.3}$$

On Γ_e, we have

$$u(R, \theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta) \tag{2.2$'$}$$

and

$$\frac{\partial^2 u(R, \theta)}{\partial\theta^2}=\sum_{n=1}^{\infty}(-n^2)(a_n\cos n\theta+b_n\sin n\theta). \tag{2.4}$$

From (2.4), we obtain the Fourier coefficients a_n, b_n $(n=1, 2, \cdots)$:

$$\begin{cases} a_n=-\dfrac{1}{\pi n^2}\displaystyle\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\cos n\varphi\, d\varphi, \\ b_n=-\dfrac{1}{\pi n^2}\displaystyle\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\sin n\varphi\, d\varphi. \end{cases} \tag{2.5}$$

And we have