THE COMPUTATIONAL COMPLEXITY OF THE RESULTANT METHOD FOR SOLVING POLYNOMIAL EQUATIONS*1)

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Abstract

Under an assumption of distribution on zeros of the polynomials, we have given the estimate of computational cost for the resultant method. The result is that, in probability $1-\mu$, the computational cost of the resultant method for finding ε-approximations of all zeros is at most

$$cd^2\left(\log d+\log\frac{1}{\mu}+\log\log\frac{1}{\varepsilon}\right),$$

where the cost is measured by the number of f-evaluations. The estimate of cost can be decreased to $c\left(d^2\log d+d^2\log\frac{1}{\mu}+d\log\log\frac{1}{\varepsilon}\right)$ by combining resultant method with parallel quasi-Newton method.

§ 1. Introduction

Generally, search algorithms such as Lehmer's or Kuhn's[1,2] only converge linearly, whereas iterative methods with high order demand an initial approximation which is sufficiently close to the zero. The resultant procedure[3] based on root-squaring process not only converges rapidly, but imposes no restriction on the initial approximation. In the light of this, we estimate the cost of the resultant method for finding all zeros in the sense of probability. To be precise, we shall prove

Main Theorem. *Suppose* $f(z)=a_0z^d+a_1z^{d-1}+\cdots+a_d$ $(a_0\neq 0)$ *is a random polynomial whose zeros are independently uniform random variables on* $[0; R]^{d}$ 2). *Then, for* $0<\varepsilon<\frac{1}{4}$, $0<\mu<\frac{1}{2}$, *in probability* $1-\mu$, *the computational cost of the resultant method for finding* ε*-approximations of all zeros is at most*

$$cd^2\left(\log_2 d+\log_2\frac{1}{\mu}+\log_2\log_2\frac{1}{\varepsilon}\right),$$

where c *only depends on* R, *and the unit of cost is defined as an* f*-evaluation which is turned into* d *multiplications and additions.*

From the theorem, we see that the cost of the method is of loglog ε type, and is a low-degree polynomial in d. As the cost is relatively low, the resultant method is tractable and worth notice.

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2) $[z; r]$ is the disk with center z and radius r.

§ 2. Elementary Assumption and Computation of Probability

Let C be a complex field. We define the following polynomial set

$$\mathfrak{P}(R)=\{f\colon \partial f=d,\ \text{all zeros}\ \zeta_i(i=1,2,\cdots,d)\ \text{of}\ f\ \text{satisfy}\ |\zeta_i|\leqslant R\} \tag{2.1}$$

as a probability space, and propose the following

Elementary Assumption The zeros $(\zeta_1, \zeta_2, \cdots, \zeta_d)$ of a polynomial equation are uniformly independent random variables on $[0, R]^d$.

Conventionally, one always supposes that the coefficient of the first term of a polynomial is 1 and other coefficients are independent uniformly random variables on $[0; R]^d$, see [4—6]. In practical computation, it is difficult to consider the distribution of the coefficients or zeros of the polynomials to be solved. The study of computational complexity is to give some information about the tractability of the method, the cost function in degree d and approximation error ε.

Assume G is a Lebesgue measurable subset of $\{(z_1, z_2, \cdots, z_d)\colon z_i\in\mathbb{C},\ |z_1|<\cdots<|z_d|,\ i=1,\cdots,d\}$, $\mathfrak{P}(G)$ is a set of the polynomials in $\mathfrak{P}(R)$ whose zero vector, after proper arrangement of the orders of components, is in G. Define the volume of G in a complex field as the volume of the relative set $\mathfrak{P}(G)$, and the probability of $\{f\in\mathfrak{P}(G)\}$ as

$$P\{f\in\mathfrak{P}(G)\}=\frac{\text{vol}\ G}{\text{vol}\ \mathfrak{P}(R)}=\frac{\text{vol}\ G}{(\pi R^2)^d/d!}.$$

Let

$$\mathfrak{D}_\rho(R)=\{f\colon f\in\mathfrak{P}(R),\ \exists\zeta_1\neq\zeta_2,\ f(\zeta_1)=f(\zeta_2)=0,\ \big||\zeta_1|-|\zeta_2|\big|<R\rho\},$$

$$\mathfrak{R}_\lambda(R)=\left\{f\colon f\in\mathfrak{P}(R),\ \exists\zeta_1\neq\zeta_2,\ |\zeta_1|<|\zeta_2|,\ f(\zeta_1)=f(\zeta_2)=0, 1-\left|\frac{\zeta_1}{\zeta_2}\right|<\lambda\right\},$$

$$\mathfrak{U}_\alpha(R)=\left\{f\colon f\in\mathfrak{P}(R),\ \exists\zeta_1,\ f(\zeta_1)=0,\ \left|\arg\zeta_1-\frac{k\pi}{2}\right|<\alpha,\ k=1\ \text{or}\ 3\right\}.$$

From the definition of polynomial sets and a simple computation of volume, the following lemmas can be proved easily.

Lemma 2.1. *Suppose* $\lambda>0$. *Then*

$$\mathfrak{D}_{R\lambda}(R)\supset\mathfrak{R}_\lambda(R).$$

Lemma 2.2.

$$P\{f\in\mathfrak{D}_\rho(R)\}\leqslant 2d^2\rho,\quad P\{f\in\mathfrak{U}_\alpha(R)\}\leqslant\frac{d\alpha}{\pi}.$$

§ 3. The Resultant Method for Solving the Polynomial Equation

Let $f(z)=a_0z^d+a_1z^{d-1}+\cdots+a_d$. Define

$$Tf(z)=f(z)\cdot f^*(z), \tag{3.1}$$

where $f^*(z)=\bar{a}_0z^d+\cdots+\bar{a}_d$. Obviously $Tf(z)$ becomes a polynomial with real coefficients. For convenience, we also write $f(z)$ for $Tf(z)$.

The resultant procedure is divided in two steps: First, compute the moduli of all zeros; second, compute all real zeros and all quadratic factors. Now we are going to give the detail.