Vol. 3 No. 2 JOURNAL OF COMPUTATIONAL MATHEMATICS April 1985

THE GENERAL NONLINEAR MUTUAL BOUNDARY PROBLEMS FOR THE SYSTEMS OF NONLINEAR WAVE EQUATIONS BY FINITE DIFFERENCE METHOD*

ZHOU YU-LIN (周毓麟)
(*Institute of Applied Physics and Computational Mathematics, Beijing, China*)

§ 1. Introduction

1. The nonlinear wave equations are often appeared in the physical, chemical, mechanical, biological, geometrical problems and others. For example, the Sino-Gordon equation

$$u_{tt}-u_{xx}=\sin u, \tag{1}$$

the nonlinear vibration equations

$$u_{tt}-u_{xx}+u^3=0, \tag{2}$$

$$u_{tt}-u_{xx}+u^5=0 \tag{3}$$

and the equation

$$u_{tt}-u_{xx}+\sinh u=0 \tag{4}$$

belong to the number of the nonlinear wave equations. A lot of works contributed to the study of the various problems of the nonlinear wave equations[1-17] and the fairly wide systems of nonlinear wave equations[18-20], such as the periodic boundary problem, Cauchy problem, first, second and other boundary problems. In the expressions of the conditions of so-called the absorbing boundary problems for the wave equation, there are the derivatives with respect to the time variable and space variable[21,22].

Let us consider in the present work the system of nonlinear wave equations of the following form

$$u_{tt}-u_{xx}+\operatorname{grad} F(u)=f(x, t, u, u_x, u_t), \tag{5}$$

which contains the above mentioned nonlinear wave equations as the simple special cases. Here $u(x, t)=(u_1(x, t), \cdots, u_m(x, t))$ is a m-dimensional vector function, $F(u)$ is a scalar function of vector variable $u\in\mathbb{R}^m$ and $f(x, t, u, p, q)$ is a m-dimensional vector function for the scalar variables x, t and the vector variables u, p, $q\in\mathbb{R}^m$. In the rectangular domain $Q_T=\{0\leqslant x<l,\ 0\leqslant t\leqslant T\}$, we take into account of the boundary problem with the nonlinear mutual boundary conditions of the form

$$\begin{aligned} u_t(0, t)&=\Phi_0(u_x(0, t), u_x(l, t), u(0, t), u(l, t), t),\\ -u_t(l, t)&=\Phi_1(u_x(0, t), u_x(l, t), u(0, t), u(l, t), t)\end{aligned} \tag{6}$$

* Received June 11, 1984.

and the initial conditions

$$\begin{aligned} u(x, 0) &= \varphi(x), \\ u_t(x, 0) &= \psi(x), \end{aligned} \tag{7}$$

where $\Phi_0(p_0, p_1, u_0, u_1, t)$ and $\Phi_1(p_0, p_1, u_0, u_1, t)$ are two m-dimensional vector functions of $t \in [0, T]$ and $u_0, u_1, p_0, p_1 \in \mathbb{R}^m$ and $\varphi(x)$ and $\psi(x)$ are two m-dimensional vector functions of $x \in [0, l]$.

We are going to construct the global generalized solution of the general nonlinear mutual boundary problem (6) and (7) for the system (5) of nonlinear wave equations by the finite difference method. The convergence of the solutions of finite difference scheme is established and the limit is the solution of the problem (6) and (7) for the system (5) of nonlinear wave equations.

By similar way we also consider the mixed problem with the boundary conditions

$$\begin{aligned} u_t(0, t) &= \Phi_0(u_x(0, t), u(0, t), t), \\ u(l, t) &= 0 \end{aligned} \tag{8}$$

and the initial conditions (7) for the system (5) of nonlinear wave equations.

We adopt the similar notations and conventions used in [23—24].

2. Suppose that for the system (5), the general nonlinear mutual boundary conditions (6) and the initial vector functions in (7), the following assumptions are valid.

(I) The scalar non-negative convex function $F(u) \geqslant 0$ of the m-dimensional vector variable $u \in \mathbb{R}^m$ is twice continuously differentiable with respect to $u \in \mathbb{R}^m$.

(II) $f(x, t, u, p, q)$ is a m-dimensional continuous in $(x, t, u, p, q) \in Q_T \times \mathbb{R}^{3m}$ vector function, continuously differentiable with respect to variable x and vector variables $u, p, q \in \mathbb{R}^m$. Further for any $(x, t) \in Q_T$ and $u, p, q \in \mathbb{R}^m$, there are

$$\begin{aligned} &|f(x, t, u, p, q)|^2,\ |f_x(x, t, u, p, q)| \leqslant A\{F(u) + |u|^2 + |p|^2 + |q|^2 + 1\}, \\ &|f_u(x, t, u, p, q)|^2 \leqslant A\{F(u) + |u|^2 + 1\}, \\ &|f_p(x, t, u, p, q)|,\ |f_q(x, t, u, p, q)| \leqslant A, \end{aligned} \tag{9}$$

where A is a constant and for brevity $|\cdot|$ denotes any components of the appropriate vector functions and any elements of the mentioned matrices.

(III) $\Phi_0(p_0, p_1, u_0, u_1, t)$ and $\Phi_1(p_0, p_1, u_0, u_1, t)$ are two m-dimensional continuously differentiable vector functions of the variable $t \in [0, T]$ and the vector variables $u_0, u_1, p_0, p_1 \in \mathbb{R}^m$. The $2m \times 2m$ Jacobi derivative matrix $\frac{\partial(\Phi_0, \Phi_1)}{\partial(p_0, p_1)}$ of the $2m$-dimensional vector function (Φ_0, Φ_1) with respect to $2m$-dimensional vector (p_0, p_1) is positively definite, i.e., there is a positive number $\sigma > 0$, such that

$$\left(\eta, \frac{\partial(\Phi_0, \Phi_1)}{\partial(p_0, p_1)} \eta\right) \geqslant \sigma |\eta|^2 \tag{10}$$

for any $2m$-dimensional vector $\eta \in \mathbb{R}^{2m}$. Furthermore, there are