ORDER INTERVAL SECANT METHOD FOR NONLINEAR SYSTEMS*

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Abstract

An order interval secant method is given. Its rate of convergence is faster than that of order interval Newton method in [1]. The existence and uniqueness of a solution to nonlinear systems and convergence of the interval iterative sequence are also proved.

Suppose $f: D\subset R^n \to R^n$, $X=[\underline{x}, \bar{x}]=\{x\in R^n \mid \underline{x}\leqslant x\leqslant \bar{x}\}\subset D$. A simple interval Newton method for testing the existence and uniqueness of solutions to the nonlinear equations

$$f(x)=0 \tag{1}$$

and an order interval Newton method for solving nonlinear equations (1), were given in [1]. This paper is to give an order interval secant method without the derivative calculation of f, which converges faster than the order interval Newton method. The convergence of this iterative method, as well as the existence and uniqueness of solutions to (1), are proved.

Notations used in this paper are the same as those in [1].

First, the definition of order convexity in [2] has to be generalized.

Definition 1. *If there exist a nonsingular matrix $P\in L(R^n)$ and $\lambda\in(0, 1)$ so that*

$$Pf(\lambda x+(1-\lambda)y)\leqslant \lambda Pf(x)+(1-\lambda)Pf(y) \tag{2}$$

for $f: D\subset R^n\to R^n$ in $X=[\underline{x}, \bar{x}]$ and for any comparable $x, y\in X$ ($x\leqslant y$ or $x\geqslant y$), then f is called P-order convex. Moreover, if

$$Pf(x)\leqslant Pf(y) \tag{3}$$

for any $\underline{x}\leqslant x\leqslant y\leqslant \bar{x}$, then f is called P-isotone convex in X. If (2) is replaced by

$$Pf(\lambda x+(1-\lambda)y)\geqslant \lambda Pf(x)+(1-\lambda)Pf(y), \tag{2'}$$

then f is called P-order concave (or P-order upper convex) in X, Moreover, if (3) is valid then f is called P-isotone concave in X.

Remark 1. If $P=I$ (the identity matrix), it is the order convexity defined by [2]. That f is P-order convex in X implies that $F=Pf$ is order convex in X.

Definition 2. *Suppose P is nonsingular, and $\underline{x}\leqslant x<y\leqslant \bar{x}$ for any two points $x=(x_1, \cdots, x_n)^T$ and $y=(y_1, \cdots, y_n)^T$ on the interval $X=[\underline{x}, \bar{x}]$. Let $F=Pf$, $A(x, y)=(a_{ij})_{n\times n}$ is called an nth-order difference matrix, where*

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$$a_{ij}=\frac{F_i\left(y-\sum_{k=0}^{j-1}(y_k-x_k)e_k\right)-F_i\left(y-\sum_{k=1}^{j}(y_k-x_k)e_k\right)}{y_j-x_j},\tag{4}$$

$$a_{i1}=\frac{F_i(y)-F_i(y-(y_1-x_1)e_1)}{y_1-x_1},\quad i=1,\ \cdots,\ n;\ j=2,\ \cdots,\ n$$

If $A_i(x, y)$ is the i-th row of matrix $A(x, y)$, by (4),

$$A_i(x,\ y)(y-x)=(a_{i1},\ \cdots,\ a_{in})\begin{pmatrix}y_1-x_1\\ \vdots\\ y_n-x_n\end{pmatrix}=F_i(y)-F_i(x),\quad i=1,\ \cdots,\ n,$$

then

$$A(x,\ y)(y-x)=F(y)-F(x).\tag{5}$$

Lemma 1. *Suppose f is P-order convex in $X=[\underline{x}, \bar{x}]$ and $F=Pf$; then, for any $\underline{x}\leqslant x<y\leqslant\bar{x}$,*

$$A(x,\ z')(y-x)\leqslant F(y)-F(x)\leqslant A(z,\ y)(y-x),\tag{6}$$

where $z=y-t(y-x)<y$, $z'=x+t(y-x)$, $t\in(0,\ 1)$, $A(x,\ z')$ and $A(z,\ y)$ are nth-order difference matriced of F defined as (4).

If f is P-order concave, then for any $\underline{x}\leqslant x<y\leqslant\bar{x}$,

$$A(x,\ z')(y-x)\geqslant F(y)-F(x)\geqslant A(z,\ y)(y-x).\tag{6'}$$

Proof. Because f is P-order convex on $X=[\underline{x}, \bar{x}]$, by (2),

$$F(z)\leqslant tF(x)+(1-t)F(y),$$

or

$$F(z)-F(y)\leqslant t(F(x)-F(y))$$

for $z=y-t(y-x)$ and $t\in(0,\ 1)$. Then

$$F(y)-F(x)\leqslant\frac{1}{t}(F(y)-F(z))=\frac{1}{t}(F(y)-F(y-t(y-x))).$$

By (4) and (5), we have

$$F(y)-F(x)\leqslant\frac{1}{t}(F(y)-F(z))=\frac{1}{t}A(z,\ y)(y-z)=A(z,\ y)(y-x).$$

Therefore, the right inequality in (6) holds. From same reason, if we set $z'=x+t(y-x)$, by (2) we have

$$F(y)-F(x)\geqslant\frac{1}{t}(F(z')-F(x))=\frac{1}{t}A(x,\ z')(z'-x)=A(x,\ z')(y-x).$$

Hence (6) holds.

(6′) can be proved similarly.

Lemma 2. *Suppose f is P-order convex in $X=[\underline{x}, \bar{x}]$ and G-differentiable, and set $z=y-t(y-x)$, $z'=x+t(y-x)$. Then, for any $\underline{x}\leqslant x<y\leqslant\bar{x}$ and $t\in(0,\ 1)$,*

(i) $$F'(x)(y-x)\leqslant F(y)-F(x)\leqslant F'(y)(y-x)\tag{7}$$

and

$$\lim_{t\to0}A(x,\ z')=F'(x),\quad \lim_{t\to0}A(z,\ y)=F'(y),$$

(ii) $$F'(x)\leqslant A(x,\ z')\leqslant A(x,\ y)\leqslant A(z,\ y)\leqslant F'(y).\tag{8}$$

Moreover, suppose $F'(x)$ and $A(x,\ y)$ are nonsingular for any $\underline{x}\leqslant x<y\leqslant\bar{x}$, $F'(x)^{-1}\geqslant0$ and $A(x,\ y)^{-1}\geqslant0$. Then

$$F'(x)^{-1}\geqslant A(x,\ z')^{-1}\geqslant A(x,\ y)^{-1}\geqslant A(z,\ y)^{-1}\geqslant F'(y)^{-1}\geqslant0\tag{9}$$