EXTRAPOLATION COMBINED WITH MULTIGRID METHOD FOR SOLVING FINITE ELEMENT EQUATIONS*1)

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Abstract

An algorithm combining the MG method with two types of extrapolation is given for solving finite element equations with any initial triangulation. A high order approximation to the solution of PDEs can be obtained at the cost of order $O(N)$ of computational work.

§ 1. Introduction

Two types of extrapolation are suggested in [1] for solving boundary value problems by successively refining meshes:

Type 1 for gaining a higher order approximation to the solution of PDEs;

Type 2 for gaining a good initial approximation in iteration.

These extrapolations are based theoretically upon the asymptotic expansion

$$u^h = u^I + c_1 h^{\alpha_1} + c_2 h^{\alpha_2} + \cdots, \quad 0<\alpha_1<\alpha_2<\cdots, \tag{1}$$

where u^h, u^I represent the discrete solution and the interpolation function of the solution of PDEs for linear finite element. It has been known that[2,3]

$$u^h(z) = u^I(z) + w(z)h^2 + O(h^2 \ln h) \tag{2}$$

holds if the solution of PDEs is smooth enough. The numerical experiments and some theoretical analysis in [4] show that asymptotic expansions also hold for less regular problems. In order to make the extrapolation of type 1 effective, the discrete solution must be accurate enough and this should cost an order of $O(N \ln N)$ of computational work for ordinary MG methods (N the number of nodes). Now an algorithm combining the MG method with type 2 extrapolation is given and its order of computational work is reduced to $O(N)$.

When we finished the paper, we learned that some authors[5,6] also worked on the same topic. But their results are limited to special regular domains and special initial partition.

§ 2. Algorithm and Analysis

Let Ω be a plane polygon. A series of nested triangulations of Ω are produced

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as follows: An initial partition Δ_0 divides Ω into a few large triangles. Then, successive midpoint refinements produce a series of partitions $\Delta_0, \Delta_1, \cdots, \Delta_k, \cdots$ with corresponding mesh sizes $h_0, h_1, \cdots, h_k, \cdots$ and $h_{k-1}=2h_k$. A series of linear finite element equations,

$$A_k u_k = f_k \tag{3}$$

corresponding to the partition Δ_k, are solved one by one. Now, an algorithm is given as follows.

1. For $k=0, 1$, solve $\tilde{u}_0 = A_0^{-1} f_0$, $\tilde{u}_1 = A_1^{-1} f_1$ directly.
2. For $k \geqslant 2$, take the initial approximation

$$u_k^0 = \Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1}) \tag{4}$$

and then perform MG iteration r times to obtain $\tilde{u}_k$.

3. If $\tilde{u}_k$ is accurate enough according to some stopping criteria such as given in [1], stop and go to do the type 1 extrapolation; otherwise go to step 2.

The MG algorithm is referred to [7]. This paper mainly deals with the initial choice of (4).

Theorem. *Let constants c_1 and c_2 satisfy, for $k=2, 3, \cdots$*

$$\rho_k \leqslant \rho < 1, \tag{5}$$

$$\|u_k - \Pi(u_{k-2}, u_{k-1})\|_{L_2(\Omega)} \leqslant c_1 h^\alpha, \quad \alpha > 0, \tag{6}$$

$$\|\Pi(u_{k-2}, u_{k-1}) - \Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1})\|_{L_2(\Omega)} \leqslant c_2 (\|u_{k-2} - \tilde{u}_{k-2}\|_{L_2(\Omega)} + \|u_{k-1} - \tilde{u}_{k-1}\|_{L_2(\Omega)}). \tag{7}$$

Constant ρ_k stands for the convergence factor of the MG iteration on Δ_k in the sense of L_2-norm. Then, when r makes $2c_2\rho^r < 1$,

$$\|u_k - \tilde{u}_k\|_{L_2(\Omega)} \leqslant c(\rho) h^\alpha, \quad k=0, 1, 2, \tag{8}$$

holds with $c(\rho) = c_1 \rho^r / (1 - c_2 \rho^r)$.

Proof. By induction. For $j=0, 1$, $u_j = \tilde{u}_j$ and (8) is trivial. Suppose now (8) is true for $j \leqslant k-1$; then, for $j=k$,

$$\begin{aligned} \|u_k^0 - u_k\|_{L_2(\Omega)} &= \|\Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1}) - u_k\|_{L_2(\Omega)} \\ &\leqslant \|\Pi(u_{k-2}, u_{k-1}) - u_k\|_{L_2(\Omega)} + \|\Pi(u_{k-2}, u_{k-1}) - \Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1})\|_{L_2(\Omega)} \\ &\leqslant c_1 h^\alpha + c_2 (\|u_{k-2} - \tilde{u}_{k-2}\|_{L_2(\Omega)} + \|u_{k-1} - \tilde{u}_{k-1}\|_{L_2(\Omega)}) \\ &\leqslant (c_1 + 2c_2 c(\rho)) h^\alpha, \end{aligned}$$

$$\|u_k - \tilde{u}_k\|_{L_2(\Omega)} \leqslant \rho^r \|u_k^0 - u_k\|_{L_2(\Omega)} \leqslant \rho^r (c_1 + 2c_2 c(\rho)) h^\alpha = c(\rho) h^\alpha.$$

The proof is thus completed.

The norm in the above theorem can be replaced by other norms as long as the corresponding (5), (6) and (7) hold.

§ 3. The Choice of Initial Approximations

Suppose that

$$u^h(z) = u^1(z) + w(z) h^2 + O(h^\tau), \quad z \in \Omega$$

with $\tau > 2$. We show how to define $\Pi(u_{k-2}, u_{k-1})$ such that (6) and (7) hold for $\alpha > 2$.