TWO-DIMENSIONAL REPRODUCING KERNEL AND SURFACE INTERPOLATION*

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Abstract

One-dimensional polynomial interpolation does not guarantee the convergency and the stability during numerical computation. For two (or multi)-dimensional interpolation, difficulties are much more raising. There are many fundamental problems, which are left open.

In this paper, we begin with the discussion of reproducing kernel in two variables. With its help we deduce a two-dimensional interpolation formula. According to this formula, the process of interpolation will converge uniformly, whenever the knot system is thickened in finitely. We have also proven that the error function will decrease monotonically in the sense of Соболев norm when the number of knot points is increased.

In our formula, knot points may be chosen arbitrarily without any request of regularity about their arrangement. We also do not impose any restriction on the number of knot points. For the case of multi-dimensional interpolation, these features may be important and essential.

§ 1. The W Space and Reproducing Kernel

1. Let

$$W=\left\{\{u|u\in C([a, b]\times[c, d]);\frac{\partial u}{\partial\xi},\ \frac{\partial u}{\partial\eta},\ \frac{\partial^2 u}{\partial\xi\partial\eta}\in L^2([a, b]\times[c, d])\right\}.$$

We define the inner product

$$(u, v)=\int_c^d d\eta\int_a^b\left(uv+\frac{\partial u}{\partial\xi}\cdot\frac{\partial v}{\partial\xi}+\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\eta}+\frac{\partial^2 u}{\partial\xi\partial\eta}\cdot\frac{\partial^2 v}{\partial\xi\partial\eta}\right)d\xi$$

and the norm

$$\|\cdot\|=(\cdot, \cdot)^{\frac{1}{2}}.$$

2. Definition. *Suppose $K_{xy}(\cdot)\in W$, $(x\in[a, b], y\in[c, d])$. If for any $u\in W$ we have*

$$u(x, y)=(u(\cdot), K_{xy}(\cdot)),$$

then we call $K_{xy}(\cdot)$ the reproducing kernel in W space.

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3. Definition.

$$\begin{cases} R_M(M') \overset{\text{def.}}{=\!=} R_{xy}(\xi, \eta) = R_x(\xi)\cdot R_y(\eta), \quad x, \xi \in [a, b], y, \eta \in [c, d], \\ R_x(\xi) = \dfrac{1}{2\,\mathrm{sh}(b-a)}[\mathrm{ch}(\xi+x-a-b)+\mathrm{ch}(|\xi-x|+a-b)], \\ R_y(\eta) = \dfrac{1}{2\,\mathrm{sh}(d-c)}[\mathrm{ch}(\eta+y-c-d)+\mathrm{ch}(|\eta-y|+c-d)]. \end{cases} \tag{1}$$

Evidently $R_{xy}\in W$, $R_{xy}(\xi, \eta) = R_{\xi\eta}(x, y)$. It is easy to verify that $R_{xy}(\xi, \eta)$ is a reproducing kernel in W space.

Proof. For any $u\in W$,

$$\begin{aligned}(u(\cdot), R_{xy}(\cdot)) &= \int_c^d d\eta \int_a^b \{u(\xi, \eta)R_{xy}(\xi, \eta) + u'_\xi(\xi, \eta)R'_{xy}(\xi, \eta)_\xi \\ &\quad + u'_\eta(\xi, \eta)R'_{xy}(\xi, \eta)_\eta + u''_{\xi\eta}(\xi, \eta)R''_{xy}(\xi, \eta)_{\xi\eta}\}d\xi \\ &= \int_c^d d\eta\Big\{R_y(\eta)\int_a^b (uR_x(\xi) + u'_\xi R'_x(\xi)_\xi)\,d\xi \\ &\quad + R'_y(\eta)_\eta \int_a^b \Big(u'_\eta R_x(\xi) + \frac{\partial}{\partial\xi}(u'_\eta)R'_x(\xi)_\xi\Big)d\xi\Big\}.\end{aligned} \tag{2}$$

Using the reproducing property of one-dimensional kernel $R_x(\xi)$[1], we have

$$\int_a^b (uR_x(\xi) + u'_\xi R'_x(\xi)_\xi)d\xi = u(x, \eta), \tag{3}$$

$$\int_a^b \Big(u'_\eta R_x(\xi) + \frac{\partial}{\partial\xi}(u'_\eta)R'_x(\xi)_\xi\Big)d\xi = u'_\eta(x, \eta). \tag{4}$$

So,

$$(u(\cdot), R_{xy}(\cdot)) = \int_c^d \{u(x, \eta)R_y(\eta) + u'_\eta(x, \eta)R'_y(\eta)_\eta\}d\eta. \tag{5}$$

Again, by the reproducing property of $R_y(\eta)$, we get

$$(u(\cdot), R_{xy}(\cdot)) = u(x, y).$$

It is also very easy to verify that $R_{xy}(\xi, \eta)$ holds many properties that $R_x(\xi)$ possesses. For instance, $R_{xy}(\xi, \eta)$ satisfies Lipschitz condition and is positively bounded both above and below[1].

§ 2. Formula of Interpolation

Let $E = [a, b]\times[c, d]$, and let $[M_i]_1^n$ be n distinct knots of interpolation on E with $M_i = (x_i, y_i)$. Denote $R_M(M') = R_{xy}(\xi, \eta)$, $M' = (\xi, \eta)$.

Put

$$\phi_k(M) = R_{M_k}(M), \quad k = 1, 2, \cdots, n. \tag{6}$$

By the reproducing property of $R_{M_k}(M)$, it follows that

$$(u(M), \phi_k(M)) = u(M_k), \quad u\in W, k = 1, 2, \cdots, n. \tag{7}$$

Evidently $\{\phi_k(M)\}_1^n$ forms a linearly independent system in W. Using Gram-Schmidt process, we can get the orthonormal system $\{\overline{\phi}_k(M)\}_1^n$:

$$\overline{\phi}_k(M) = \sum_{i=1}^k \beta_{ki}\phi_i(M), \quad k = 1, 2, \cdots, n. \tag{8}$$

In order to describe the degree of thickness of the knot system, we introduce a