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HIGHLY ACCURATE NUMERICAL SOLUTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS ON GENERAL REGIONS*

ZHOU GUO-FU (周国富)
(Institute of Mathematical Sciences, Chengdu Branch, Academia Sinica, Chengdu, China)

Abstract

We prove that Lin Qun and Lu Tao's splitting extrapolation method and correction method can be effectively applied to raise the accuracy of the numerical solution of elliptic boundary value problems on general regions, i.e., to obtain approximate solutions with fourth- or fifth-order precision in the maximum norm.

§ 1. Introduction

The solution of the linear elliptic boundary value problem

$$\begin{cases} Lu(x) \equiv \sum_{j=1}^{N}\left[a_j(x)\dfrac{\partial^2 u(x)}{\partial x_j^2} + b_j(x)\dfrac{\partial u}{\partial x_j}\right] + d(x)u(x) = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega \end{cases} \tag{1}$$

by the finite difference method has a long history, see, e.g., [1] and rferences therein. It is well-known that there are many methods, based on the asymptotic expansion, for acceleration of convergence, but the one for problem (1) was obtained by Böhmer[1] only recently.

In this paper, we first give a simpler proof of Böhmer's result. Then we explain how to obtain solutions with fourth- or fifth-order precision by the splitting extrapolation method[7]. Finally, based on the idea in [8], we formulate the correction analogue for (1) and prove that the approximate solution has fourth-order precision.

The splitting extrapolation method[7] can save much computational work and storage in comparison with the usual extrapolation along all variables, for it is essentially equal to the procedure in which the one-dimensional extrapolation is done N times, where N is the dimension of problem (1). Moreover, it is appropriate for parallel computers. The correction method has the advantage that to obtain a more accurate solution, one does not need to solve the original discrete problem on a smaller mesh, but to solve another discrete problem on the original mesh, which is easier.

§ 2. Formulation of Difference Analogue

Let us consider the numerical solution for (1) in which Ω is an arbitrary

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bounded and connected region in the N-dimensional Euclidean space R^N. To define the analogue, we first introduce the following notations

e_j, j-th unit vector in the j-th co-ordinate direction,

$h>0$, the step, $\mathbb{N}=\{1, 2, \cdots, N\}$,

$\mathbb{Z}=\{0, \pm 1, \pm 2, \cdots\}$,

$N_h=\Omega\cap\{x\in R^N:\ x_j=hn_j,\ n_j\in\mathbb{Z},\ \forall j\in\mathbb{N}\}$,

$\Omega_h=\{x\in N_h:\ x\pm he_j\in\Omega,\ \forall j\in\mathbb{N}\}$,

$\Omega'_h=N_h-\Omega_h$.

For every $x\in\Omega'_h$, by notations we know that there exist a non-empty subset $I=I^+\cup I^-$ of $\mathbb{N}$ such that $x+he_j\bar{\in}\Omega$, $\forall j\in I^+$, $x-he_j\bar{\in}\Omega$, $\forall j\in I^-$ and $x\pm he_j\in\Omega$, $\forall j\in \mathbb{N}-I$. We assume that $\partial\Omega$ is smooth enough and h is so small that we have $x-(v-1)he_j\in\Omega$, $v=2, \cdots, k$, $\forall j\in I^+$ and $x+(v-1)he_j\in\Omega$, $v=2, \cdots, k$, $\forall j\in I^-$. Moreover, for any $j\in I^\pm$, there is exactly one intersection point $x\pm(1-s_j^\pm)he_j$, $0\leqslant s_j^\pm<1$, of the line segment from x to $x\pm he_j$ with $\partial\Omega$. Taking $x\in\Omega'_h$, we let $\partial\Omega_h$ be the set consisting of the intersection points.

Now, we define an operator L_h: $F_h\to F_h$, where $F_h=\{u_h:\ N_h\cup\partial\Omega_h\to R\}$. For every $x\in\Omega_h$, let

$$\begin{aligned} L_hu_h(x)=&\frac{1}{h^2}\sum_{j=1}^N a_j(x)\left[u_h(x+he_j)-2u_h(x)+u_h(x-he_j)\right]\\ &+\frac{1}{2h}\sum_{j=1}^N b_j(x)\left[u_h(x+he_j)-u_h(x-he_j)\right]+d(x)u_h(x). \end{aligned}\tag{2.1}$$

For every $x\in\partial\Omega_h$, $L_hu_h(x)=u_h(x)$. Finally, for every $x\in\Omega'_h$, we still use (2.1) to define $L_hu_h(x)$. To this end, noting that $x\pm he_j\bar{\in}\Omega$, $\forall j\in I^\pm$, we replace $u_h(x\pm he_j)$ in (2.1) by

$$u_h(x\pm he_j)=\alpha_{j,-1}^\pm u_h(x\pm(1-s_j^\pm)he_j)+\sum_{v=0}^{k-1}\alpha_{j,v}^\pm u_h(x\mp vhe_j),\tag{2.2}$$

where

$$\alpha_{j,-1}^\pm=k!\Big/\prod_{l=1}^k(l-s_j^\pm),\quad \alpha_{j,v}^\pm=(-1)^{v+1}s_j^\pm/(v+1-s_j^\pm),\tag{2.3}$$

which is meaningful from the assumptions. Thus, L_h is well defined and we obtain the discretization problem corresponding to the continuous problem (1), i.e., the difference analogue

$$\begin{cases} L_hu_h(x)=f(x), & x\in\Omega_h\cup\Omega'_h,\\ u_h(x)=g(x), & x\in\partial\Omega_h. \end{cases}\tag{2.4}$$

It should be pointed out that the discrete problem (2.4) is essentially the same as that in [1].

§ 3. A Priori Estimate for the Discrete Problem

In this section we rewrite the results of Bramble-Hubbard[3,4] in a more convenient form for our use.

We express the operator L_h as