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THE SPECTRAL METHOD FOR SYMMETRIC REGULARIZED WAVE EQUATIONS*

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§1. Introduction

A symmetric version of the regularized long wave equations (SRLWE)

$$\left(\frac{\partial^2}{\partial x^2}-1\right)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\rho+\frac{u^2}{2}\right),\qquad (1.1)$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial x}=0$$

has been investigated in [1]. The system (1.1) of equations is shown to describe weakly nonlinear ion acoustic and space-change waves. The hyperbolic secant squared solitary waves, four invariants and the numerical results have been obtained in [1]. Obviously, eliminating ρ in (1.1), we get a class of RLWE

$$u_{tt}-u_{xx}+\left(\frac{1}{2}u^2\right)_{xt}-v_{xxtt}=0.\qquad (1.2)$$

Replacing the derivative for t with the derivative for x in the third and the fourth terms of (1.2), we get the Boussinesq equation. In this note we consider the periodic initial value problem for generalized nonlinear wave equations (including (1.1))

$$\begin{cases} u_t-u_{xxt}+\rho_x+f(u)_x=g(u,\rho,u_x), & (1.3)\\ \rho_t+u_x=h(\rho), & (1.4)\\ u|_{t=0}=u_0(x),\ \rho|_{t=0}=\rho_0(x),\quad -\infty<x<\infty, & (1.5)\\ u(x-\pi,t)=u(x+\pi,t),\ \rho(x-\pi,t)=\rho(x+\pi,t),\ -\infty<x<\infty,\ t\geq 0, & (1.6)\end{cases}$$

where $u(x,t)$, $\rho(x,t)$ are unknown real functions, and $f(u)$, $h(\rho)$ are known real functions. We propose the spectral method (continued and discrete) for the problem (1.3)—(1.6), establish the error estimates and convergence for the approximate solution, and prove the existence and uniqueness of the classical smooth solution for the system (1.3)—(1.6).

§2. Continued Spectral Method and Priori Estimates

First we introduce some spaces and notations. Let $C^l(\Omega)=C^l([-\pi,\pi])$ denote the space of functions, l times continuously differentiable over the interval $[-\pi,\pi]$. $L_p(\Omega)$ denotes the Lebesque space of measurable functions $u(x)$ with p-th power absolute value $|u|$ integrable over the interval $[-\pi,\pi]$ with the norm

$$\|u\|_{L_p}=\left(\int_{-\pi}^{\pi}|u|^p dx\right)^{\frac{1}{p}}.$$

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If we define the inner product

$$(u, v)=\int_{-\pi}^{\pi} u(x)v(x)dx, \quad \|u\|^2_{L_2}=(u, u),$$

then $L_2[-\pi, \pi]$ is a Hilbert space.

Let $L_\infty(\Omega)$ denote the Lebesque space of measurable functions $u(x)$ over the interval $[-\pi, \pi]$, which are essentially bounded, with the norm

$$\|u\|_{L_\infty}=\operatorname{ess\,sup}_{x\in\Omega} |u(x)|.$$

Let $H^l(\Omega)$ denote the space of the functions with generalized derivatives $D^s u(|s|\leqslant l)$ with the norm $\|u\|^2_l=\sum_{|s|\leqslant l}\|D^s u\|^2_{L_2}$. $L^\infty(0, T; H^l)$ denotes the space of the functions $u(x, t)$ which belong to H^l as a function of x for every fixed $t(0\leqslant t\leqslant T)$ and $\sup_{0\leqslant t\leqslant T}\|u(\cdot, t)\|_l<\infty$. Especially,

$$\|u\|_{L^\infty(0,T;L_2)}=\sup_{0\leqslant t\leqslant T}\|u\|_{L_2} \quad \text{or} \quad \|u\|_{L_2\times L_\infty}.$$

Let $V^l=\{u\in H^l(\Omega)\,|\,u^j(x-\pi)=u^j(x+\pi),\ 0\leqslant j\leqslant l-1\}$ be a periodic functional space, where $u^j=\frac{d^j u}{dx^j}$,

$$\|u\|^2_V=\|u\|^2_{L_2}+\left\|\frac{du}{dx}\right\|^2_{L_2}, \ V\in H^1, \ H=L_2.$$

For the backward difference quotient of $u(x, t)$ for t, we employ the following notation

$$u_{\bar t}(x, t)=\frac{1}{\Delta t}[u(x, t)-u(x, t-\Delta t)].$$

Let F_k denote the projection from H to $H_k=\operatorname{span}(v_{-k}, \cdots, v_k)$,

$$F_k g=\sum_{j=-k}^{k}(g, v_j)v_j,$$

where $v_j=\frac{1}{\sqrt{2\pi}}e^{ijx}$, $i=\sqrt{-1}$.

Set $R_k g=g-F_k g$, when $k\to\infty$, $R_k g\to 0$. From the Bessal inequality, we have

$$\|F_k g\|_{L_2}\leqslant\|g\|_{L_2}, \quad g\in H=L_2, \tag{2.1}$$

and Bernstein's estimate[9].

Suppose that the periodic function $g(x)$ is k $(k\geqslant 1)$ times differentiable and the k-th derivative is bounded, i.e.,

$$|g^{(k)}(x)|\leqslant M_k. \tag{2.2}$$

Then there exists a positive constant A, such that

$$|R_n g|\leqslant AM_k \log n/n^k, \quad n\geqslant 2. \tag{2.3}$$

In this section, we consider the continued spectral method. We construct the approximate solutions of the problem (1.3)—(1.6) as follows

$$\begin{aligned} u_k(\cdot, t)=u_k(t)&=\sum_{j=-k}^{k}\alpha_{jk}(t)v_j(x),\\ \rho_k(\cdot, t)=\rho_k(t)&=\sum_{j=-k}^{k}\beta_{jk}(t)v_j(x). \end{aligned} \tag{2.4}$$

The coefficient functions $\alpha_{jk}(t)$, $\beta_{jk}(t)$ should satisfy the equations