AN EXACT SOLUTION TO LINEAR PROGRAMMING USING AN INTERIOR POINT METHOD*

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Abstract

This paper presents sufficient conditions for optimality of the Linear Programming (LP) problem in the neighborhood of an optimal solution, and applies them to an interior point method for solving the LP problem. We show that after a finite number of iterations, an exact solution to the LP problem is obtained by solving a linear system of equations under the assumptions that the primal and dual problems are both nondegenerate, and that the minimum value is bounded. If necessary, the dual solution can also be found.

§ 1. Introduction

As is well-known, an optimal solution to the linear programming (LP) problem is obtained at an extreme point of the feasible region under the assumptions that the primal and dual problems are both nondegenerate, and that the minimum value of the LP problem is bounded. Hence, as long as nonbasic variables can be distinguished in the neighborhood of an optimal solution, an exact solution can be found by solving a linear system of equations. This idea will be exploited by use of an interior point method.

This paper presents sufficient conditions for optimality of the LP problem and applies them to an interior point method to obtain an exact solution. First of all, a second order estimate of the dual solutions and Lagrange multipliers in the neighborhood of an optimal solution are given for a standard form of the LP problem under the following assumptions: the primal and dual problems are both nondegenerate, and the minimum value of the LP problem is bounded. Then a vector of quasi-Lagrange multipliers (QLM) is introduced in order to set up the formula to estimate the value of Lagrange multipliers. The nonbasic variables will be discarded based on the estimated value of the Lagrange multipliers, and an exact solution of the LP problem will be achieved by solving a linear systems of equations. The dual solutions can also be obtained at the same time if necessary.

Section 2 shows how to set up the formula of the second-order estimate of the dual solution and Lagrange multiplier vector in the neighborhood of an optimal solution. Section 3 describes the proposed interior point method for solving the LP problem in terms of an affine transformation, and discusses the method's convergence. Section 4 applies the results described in Sections 2 and 3 to the interior point method, and shows that an exact optimal solution is obtained in a finite number of iterations.

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§ 2. Optimality Conditions

This section presents the sufficient conditions for optimality of the LP problem in the neighborhood of an optimal solution. A second order estimation formula of the dual solution and Lagrange multipliers is given under certain assumptions that a strictly interior feasible point in the neighborhood of an optimal solution.

First of all, we consider the following standard form of a linear program (LP):

$$\text{Minimize } z=c^T x, \tag{2.1}$$

$$\text{Subject to } Ax=b, \tag{2.2}$$

$$x\geqslant 0, \tag{2.3}$$

where A is an $m\times n$ real matrix with rank m, $m<n$, b and c are real vectors in R^m and R^n, respectively, and x is a real variable in R^n.

Suppose that the LP problem is feasible and nondegenerate. Assume that $\bar{x}$ is a strictly interior feasible point satisfying (2.2)—(2.3). Then, it can be expressed as

$$\bar{x}=x_v+\bar{x}-x_v=x_v+u, \tag{2.4}$$

where x_v is a basic feasible solution of the LP problem, and

$$x_v=\begin{bmatrix} x_B \\ 0 \end{bmatrix}, \tag{2.5}$$

$$u=\begin{bmatrix} u_B \\ u_N \end{bmatrix}=\bar{x}-x_v. \tag{2.6}$$

Let $N(x, \delta)$ denote the Euclidean ball about x of radius δ, and D, D_1, D_2, D_3 denote the $n\times n$, $m\times m$, $m\times m$, and $(n-m)\times(n-m)$ diagonal matrices, containing the components of $\bar{x}$, x_B, u_B, u_N, respectively. Then

$$D=\begin{bmatrix} D_1+D_2 & 0 \\ 0 & D_3 \end{bmatrix}. \tag{2.7}$$

In order to describe the main results in this section, we define vectors $\bar{y}$, c_p, and $\bar{v}$, by

$$\bar{y}=(AD^2A^T)^{-1}AD^2c, \tag{2.8}$$

$$c_p=Dc-DA^T\bar{y}, \tag{2.9}$$

$$\bar{v}=D^{-1}c_p. \tag{2.10}$$

The vector $\bar{v}$ contains the quasi-Lagrange multipliers.

Theorem 2.1. *Suppose that the* LP *problem is feasible and nondegenerate. Assume that x^* is its unique optimal solution, and y^* is the corresponding dual optimal solution. If $\bar{x}$ is a strictly interior feasible point, such that $\bar{x}\in N(x^*, \delta)$, then $\bar{y}$ is a second-order estimate of the dual solution y^*, where δ is a sufficiently small positive scalar.*

Theorem 2.2. *Suppose that the* LP *problem is feasible and nondegenerate. Assume that x^* is its unique optimal solution, and v^* is the vector of Lagrange multipliers with respect to x^*. If $\bar{x}$ is a strictly interior feasible point, such that $\bar{x}\in N(x^*, \delta)$, then the vector $\bar{v}$ is a second order estimate of v^*, where δ is a sufficiently small positive scalar.*

To prove these theorems, we introduce several lemmas.

Lemma 2.3. *Suppose that the* LP *problem is feasible and nondegenerate. Assume that $\bar{x}$ is a strictly interior feasible point satisfying* (2.2)—(2.3). *Then*