Vol. 5 No. 1 JOURNAL OF COMPUTATIONAL MATHEMATICS January 1987

ERROR ANALYSIS OF LOCAL REFINEMENTS OF POLYGONAL DOMAINS*[1)]

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Abstract

This paper gives a thorough analysis of the local refinement method on plane polygonal domains, with special attention to the treatment of reentrant corner. Convergence rates of the finite element method under various norms are derived via a systematic treatment of the interpolation theory in weighted Sobolev spaces. It is proved that by refining the mesh suitably, the finite element approximations for problems with singularities achieve the same convergence rates as those for smooth solutions.

§ 1. Introduction

To the authors knowledge, analysis of the local refinement method was initiated by A. Schatz and L. Wahlbin. Based on an asymptotic expression of the solution near a corner, which reveals the singularities of the solution, they obtained in [4] some error estimates on locally refined meshes. But from the point of view of approximation theory, their results are on functions of particular forms, rather than the norms of the error operator in suitable spaces. By introducing weighted Sobolev spaces, [1] gave a satisfactory answer to this problem. They also obtained the inverse estimates which show that their results cannot be improved. Unfortunately, the techniques used there cannot be extended readily to the case of high order elements. In this paper, by using a systematic treatment of the interpolation theory in weighted Sobolev spaces, we give a thorough analysis of the finite element method on locally refined meshes. Estimates obtained indicate the dependence of the approximation properties on the singularities of the solution, the order of the elements and the degree of refinement.

We use conventional notations in this paper, with C as a generic constant, which may assume different value in different places.

§ 2. Weighted Sobolev Spaces

Let Ω be a plane polygonal domain, with vertices $x_1, x_2, \cdots, x_M$ (see Fig. 1), ω_j is the inner angle of Ω at x_j, and $\alpha=\frac{\pi}{\omega_j}$,

$$V=\{x_1, x_2, \cdots, x_M\}.$$

* Received April 8, 1986.

1) Projects supported by the Science Fund of the Chinese Academy of Sciences.

For $\beta=(\beta_1, \beta_2, \cdots, \beta_M)$, define

$$\Phi_\beta(x)=\prod_{i=1}^{M}|x-x_i|^{\beta_i},$$

where $|\cdot|$ is the Euclidean norm in R^2.

Let $\beta+j-2=(\beta_1+j-2, \cdots, \beta_M+j-2)$ and likewise $\beta j=(\beta_1 j, \cdots, \beta_M j)$.

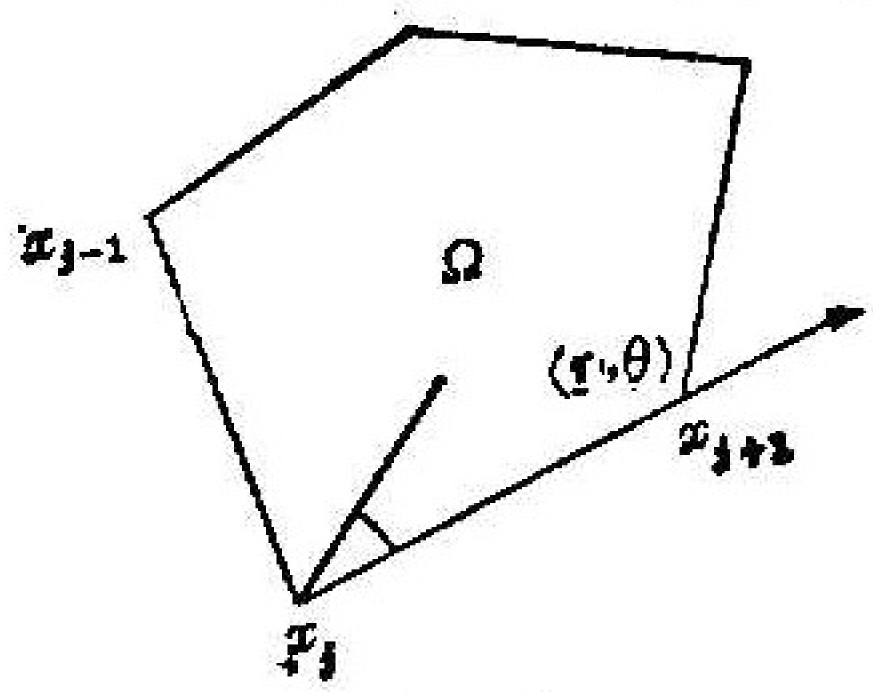

Fig. 1

For two vectors $\beta=(\beta_1, \cdots, \beta_M)$, $\gamma=(\gamma_1, \cdots, \gamma_M)$, we say $\beta>\gamma$, if $\beta_j>\gamma_j$, for all j, $1\leqslant j\leqslant M$. If, for instance, γ is real, then $\beta>\gamma$ is understood as $\beta_j>\gamma$, for all j, $1\leqslant j\leqslant M$.

Let $H_\beta^m(\Omega)$ be the complement of $C^\infty(\bar{\Omega})$ under the norm:

$$\|u\|_{m,\beta,\Omega}=\|u\|_{L^2(\Omega)}+|u|_{m,\beta,\Omega},$$

where

$$|D^j u|=\sum_{\alpha_1+\alpha_2=j}\left|\frac{\partial^j u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right|, \quad |u|_{m,\beta,\Omega}=\left(\int_\Omega \Phi^2_{\beta+m-2}|D^m u|^2\,dx\right)^{1/2}.$$

If $\beta<1$, then by using Lemma 4.9 of [3], we have

$$H_\beta^m(\Omega)\subsetneq H_\beta^1(\Omega)\subsetneq H^1(\Omega).$$

This important property will be used in proving Theorem 1.

§ 3. Interpolation Theory

Denote by d_T the diameter of a triangle T. For $\gamma=(\gamma_1, \cdots, \gamma_M)$, as in [1], we define a triangulation Δ to be of (h, θ_0, γ)-type, if

(1) Δ satisfies the conventional requirements on triangulations (see, e.g. [2], p. 51).

(2) (Zlamal's condition) For any $T\in\Delta$, if θ is an interior angle of T, then $\theta\geqslant\theta_0>0$.

(3) If $T\cap V=\emptyset$, then $C_0(\theta_0)\Phi_\gamma(x)h\leqslant d_T\leqslant C_1(\theta_0)\Phi_\gamma(x)h$, $\forall x\in T$.

(4) If $T\cap V\neq\emptyset$, then $C_0(\theta_0)\sup\limits_{x\in T}\Phi_\gamma(x)h\leqslant d_T\leqslant C_1(\theta_0)\sup\limits_{x\in T}\Phi_\gamma(x)h$.

Here, $C_0(\theta_0)$, $C_1(\theta_0)$ are constants depending only on θ_0.

If $\gamma_j>0$, the triangulation is no longer quasi-uniform. If we use d_j to denote the diameter of an element near the corner x_j, then

$$d_j\sim h^{1/(1-\gamma_j)}.$$

To define the interpolation, we need the following result, which can be easily proved by using the corresponding result of [1] and Lemma 4.9 of [3].

Lemma 1. *If* $0\leqslant\beta<1$, $m\geqslant2$, *then* $H_\beta^m(\Omega)\subsetneq C^0(\bar{\Omega})$.

Denote by $\varphi_{m-1}(\Delta)$ $(m\geqslant2)$ the space of continuous functions which are piecewise polynomials of degree $m-1$ or less on Δ. Because of Lemma 1, the Lagrangian interpolation operator

$$u\mapsto u_I$$

is well defined on $H_\beta^m(\Omega)$: $H_\beta^m(\Omega)\to\varphi_{m-1}(\Delta)$.

To estimate the interpolation error, we need the following theorem.