PERTURBATION ANALYSIS FOR SOLUTIONS OF ALGEBRAIC RICCATI EQUATIONS*

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Abstract

This paper discusses the conditioning of algebraic Riccati equations, i. e. the influence of perturbations in data on the positive semi-definite solution. A perturbation bound for the solution is given.

Notation. The symble $\mathbb{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices, and $\mathbb{C}^n=\mathbb{C}^{n\times 1}$. $\|\cdot\|_2$ denotes the spectral norm and the Euclidean vector norm. The superscript H is for conjugate transpose. $A\geqslant 0$ means that matrix A is positive semi-definite. $\lambda(A)$ denotes the spectrum of a matrix A. I_n denotes the n-th order identity matrix. Re λ denotes the real part of a complex number λ.

§ 1. Introduction

Algebraic Riccati equations arise in optimal control applications. The algebraic Riccati equation for continuous-time systems takes the form

$$A^HX+XA-XNX+K=0, \tag{1.1}$$

where $A, N, K\in\mathbb{C}^{n\times n}$, $N^H=N\geqslant 0$, $K^H=K\geqslant 0$. The positive semi-definite solution $X=X^H\geqslant 0$ of (1.1) is required.

Let $N=BB^H$ and $K=C^HC$ be full-rank factorizations of N and K, respectively. Under the assumption that (A, B) is stabilizable and (C, A) is detectable, (1.1) is known to have a unique positive semi-definite solution X, and $A-NX$ is stable.

Definition 1.1. *$M\in\mathbb{C}^{2n\times 2n}$ is said to be Hamiltonian if $J^{-1}MJ=-M^H$, where* $J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$.

Now consider the Hamiltonian matrix

$$M=\begin{pmatrix}A & N\\ K & -A^H\end{pmatrix}. \tag{1.2}$$

Under the assumption above, the eigenvalues of M have nonzero real part. If $\begin{pmatrix}U_1\\ U_2\end{pmatrix}$ is a $2n\times n$ matrix such that $M\begin{pmatrix}U_1\\ U_2\end{pmatrix}=\begin{pmatrix}U_1\\ U_2\end{pmatrix}S$, where S is stable, U_1 is invertible and $X=-U_2U_1^{-1}$ is the positive semidefinite solution of (1.1).

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The conditioning of algebraic Riccati equations, i. e. the influence of perturbations in the data on the solution, was studied to some extent in [3], [4] and [7]. [5] pointed out that it is still an open problem.

By using the perturbation theorem of invariant subspaces of a matrix, [7] obtained some useful results. This paper will continue the discussion on this problem.

§ 2. The Separation of a Stable Matrix

In [6] the separation of two matrices is defined and denoted by $\text{sep}(A, B)$. Now we introduce the following definition.

Definition 2.1. *Let $A\in\mathbb{C}^{n\times n}$. The separation of A is the number* $\text{sep}(A)$ *defined by*

$$\text{sep}(A)=\inf_{\substack{P^H=P\\ \|P\|=1}}\|PA+A^HP\|, \tag{2.1}$$

where $\|\cdot\|$ denotes any consistent norm on $\mathbb{C}^{n\times n}$.

In particular, when the norm in (2.1) is taken to be the spectral norm and Frobenious norm, it is denoted by $\text{sep}_2(A)$ and $\text{sep}_F(A)$, respectively.

By [6], it is easy to prove that $\text{sep}(A)$ has the following properties:

Property 1. Let $A, X\in\mathbb{C}^{n\times n}$ with X nonsingular. Then

$$\text{sep}(X^{-1}AX)\geqslant\frac{\text{sep}(A)}{\varkappa(X)\varkappa(X^H)},$$

where $\varkappa(X)=\|X\|\|X^{-1}\|$. If X is unitary, then

$$\text{sep}_P(X^HAX)=\text{sep}_P(A),\ P=2,\ F.$$

Property 2. Let $A, E\in\mathbb{C}^{n\times n}$. Then

$$\text{sep}(A)-(\|E\|+\|E^H\|)\leqslant\text{sep}(A+E)\leqslant\text{sep}(A)+(\|E\|+\|E^H\|).$$

Property 3. Let $A\in\mathbb{C}^{n\times n}$ with $\lambda(A)=\{\lambda_i: i=1, 2, \cdots, n\}$. Then

$$\text{sep}_P(A)\leqslant 2\min_{1\leqslant i\leqslant n}|\text{Re}\,\lambda_i|,\ P=2, F.$$

On that basis, we will give a further discussion on the property of the separation of a stable matrix. Let $A\in\mathbb{C}^{n\times n}$ be a stable marix with $\lambda(A)=\{\lambda_i(A): i=1, 2, \cdots, n, |\text{Re}\,\lambda_1(A)|\geqslant\cdots\geqslant|\text{Re}\,\lambda_n(A)|\}$. If P is Hermitian, write $\lambda(P)=\{\lambda_i(P): i=1, 2, \cdots, n, |\lambda_1(P)|\geqslant\cdots\geqslant|\lambda_n(P)|\}$.

It is easy to prove the following lemma.

Lemma 2.1. *Let $H\in\mathbb{C}^{n\times n}$ be Hermitian. Then*

$$\|H\|_2=\max_{\substack{x\in\mathbb{C}^n\\ \|x\|_2=1}}|x^HHx|.$$

In addition, if the signs of eigenvalues of H are the same, then

$$|\lambda_n(H)|=\min_{\substack{x\in\mathbb{C}^n\\ \|x\|_2=1}}|x^HHx|.$$

By Lemma 2.1, we can estimate a lower bound of the separation of a stable matrix.

Theorem 2.1. *Let $A\in\mathbb{C}^{n\times n}$ be stable.*

(1) *If A is normal, then*