FINITE ELEMENT ANALYSIS FOR A CLASS OF SYSTEM OF NONLINEAR AND NON-SELF-ADJOINT SCHRÖDINGER EQUATIONS*

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Abstract

Because the nonlinear Schrödinger equation is met in many physical problems and is applied widely, the research on well-posedness for its solution and numerical methods has aroused more and more interest. The self-adjoint case has been considered by many authors [1-5]. For a class of system of nonlinear and non-self-adjoint Schrödinger equations which refers to excitons occurring in one dimensional molecular crystals and in α spiral biomolecules, Guo Boling studied in [6], the pure initial and periodic initial value problems of this system and obtained the existence and uniqueness of its solution. In [7] we discussed the difference solution of this system and obtained its error estimate. In this paper, we shall study the finite element method for the periodic initial value problem of this system. Just as Guo pointed out in [6], since it has a non-self-adjoint term, it not only brings about trouble in mathematics, but also creates more difficulty in numerical analysis.

Our analysis will show that for this system, in theoretically we can obtain the same results as when it has no non-self-adjoint term.

§1. Notations and Statement of the Problem

Write $I=[0,\ 2\pi]$. Let $L_2=L_2(I)$ denote the set of complex valued functions which are square integrable. The scalar product of f and g in $L_2(I)$ is denoted by $(f,\ g)=\int_I f\bar{g}\,dx$ and the norm of f by $\|f\|_{L_2}=\left(\int_I |f|^2dx\right)^{\frac{1}{2}}$; $L_\infty=L_\infty(I)$ denotes the set of essentially bounded complex value functions, the norm is defined as $\|f\|_{L_\infty}=\operatorname{esssup}_{x\in I}|f(x)|$, and $\widetilde{H}^r(I)$ denotes the space of complex value periodic functions, which have square integrable generalized derivatives $D^kf(x)$ $(k\leqslant r)$. Let $\|f\|_r=(\sum_{k\leqslant r}\|D^kf\|^2_{L_2})^{\frac{1}{2}}$ denote the norm of f in $\widetilde{H}^r(I)$. The space $L^\infty(0,\ T;\ \widetilde{H}^r)$ consists of complex value functions $u(x,\ t)$ that, as a function of x, belong to $\widetilde{H}^r$ and that satisfy $\sup_{0<t<T}\|u(\cdot,t)\|_r<\infty$. In particular, if $r=0$, it is L^∞ $(0,\ T;\ \widetilde{L}_2)$, where $\widetilde{L}_2$ is the set of periodic functions in L_2.

Now we consider the following periodic initial value problem for the system of nonlinear and non-self-adjoint Schrödinger equations

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$$\begin{cases} i\dfrac{\partial \boldsymbol{u}}{\partial t}-\dfrac{\partial}{\partial x}\left(A(x)\dfrac{\partial \boldsymbol{u}}{\partial x}\right)+B\dfrac{\partial \bar{\boldsymbol{u}}}{\partial x}+\beta(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}+K(x,t)\boldsymbol{u} \\ \quad =\boldsymbol{G}(x,t), \quad -\infty<x<+\infty,\ 0<t<T, \\ \boldsymbol{u}(x,0)=\boldsymbol{u}^0(x), \quad -\infty<x<+\infty, \\ \boldsymbol{u}(x+2\pi,t)=\boldsymbol{u}(x,t), \quad 0<t<T, \end{cases} \tag{1.1}$$

where

$$A(x)=\begin{pmatrix} a_1(x) & & & \\ & a_2(x) & & \\ & & \ddots & \\ & & & a_N(x) \end{pmatrix}, \quad B=\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{pmatrix},$$

$$K(x,t)=\begin{pmatrix} K_{11}(x,t), & K_{12}(x,t), & \cdots, & K_{1N}(x,t) \\ K_{21}(x,t), & K_{22}(x,t), & \cdots, & K_{2N}(x,t) \\ \cdots & \cdots & \cdots & \cdots \\ K_{N1}(x,t), & K_{N2}(x,t), & \cdots, & K_{NN}(x,t) \end{pmatrix},$$

$\boldsymbol{u}=\boldsymbol{u}(x,t)=(u_1(x,t), u_2(x,t),\cdots, u_N(x,t))^T$ is an unknown complex vector value function, $\bar{\boldsymbol{u}}$ is the conjugate of $\boldsymbol{u}$, $|\boldsymbol{u}|^2=\sum_{l=1}^{N}|u_l|^2$, $\boldsymbol{G}(x,t)=(G_1(x,t), G_2(x,t), \cdots, G_N(x,t))^T$, $G_l(x,t)$ are known complex value functions; $a_l(x)$, $\beta(x)$, $q(s)$, $K_{lr}(x,t)$ are known real value function, $\boldsymbol{u}^0(x)=(u_1^0(x), u_2^0(x), \cdots, u_N^0(x))^T$ is a given complex vector value function; α_l are constants. Besides, these functions satisfy the following conditions:

$$\begin{aligned} &0<m\le a_l(x)\le M, \ |a_l(x)|,\ |a_l'(x)|\le K; \\ &K_{lr}=K_{rl};\ |K_{lr}(x,t)|,\ \left|\frac{\partial K_{lr}}{\partial t}\right|\le K; \\ &\sup_{0<t<T}\int_I |G_l(x,t)|^2dx\le G,\ \left|\frac{\partial G_l}{\partial t}\right|\le G;\ l,r=1,2,\cdots,N, \\ &|q(s)|\le As,\ s>0,\ |\beta(x)|\le B_0, \end{aligned} \tag{A}$$

where m, M, $\cdots$, G are positive constants.

By the results of [6], we know that (1.1) has solution $\boldsymbol{u}(x,t)$. Moreover $\boldsymbol{u}(x,t)\in L^\infty(0,T;\tilde{H}^2)$, $\boldsymbol{u}_t(x,t)\in L^\infty(0,T;\tilde{L}_2)$ and $\boldsymbol{u}(x,t)$ satisfies

$$\begin{cases} i(u_{lt},v)+\left(a_l\dfrac{\partial u_l}{\partial x},\dfrac{\partial v}{\partial x}\right)+\left(\alpha_l\dfrac{\partial \bar{u}_l}{\partial x},v\right)+(\beta(x)q(|\boldsymbol{u}|^2)u_l,v)+\sum_{j=1}^{N}(K_{lj}u_j,v) \\ \quad =(G_l(\cdot,t),v), \\ (u_l(\cdot,0),v)=(u_l^0,v),\ \forall v\in\tilde{H}^2,\ l=1,2,\cdots,N. \end{cases} \tag{1.2}$$

In the following, we write $A_l(u,v)=\int_I a_l(x)\dfrac{\partial u}{\partial x}\dfrac{\partial \bar{v}}{\partial x}dx$, and $A(\boldsymbol{u},\boldsymbol{v})=\sum_{l=1}^{N}A_l(u_l,v_l)$.

§ 2. The Semi-Discrete Finite Element Method and Its Error Estimation

We adopt the finite element method to find the approximate solution of (1.2).