THE DISCRETE-TIME FINITE ELEMENT METHODS FOR NONLINEAR HYPERBOLIC EQUATIONS AND THEIR THEORETICAL ANALYSIS*

YUAN YI-RANG (袁益让) WANG HONG (王 宏)
(*Shandong University, Jinan, China*)

Abstract

In this paper, some discrete-time finite element methods for nonlinear hyperbolic equations are derived and their theoretical analysis is given. The stability and the convergence of the finite element method for linear and semi-linear hyperbolic equations have already been discussed[1-3].

Consider the initial boundary problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - [\nabla_x, \mathscr{A}(x, u)\nabla_x u] = B(x, u)\nabla_x u + f(x, u), & x\in\Omega,\ t\in[0, T], \\ \dfrac{\partial u}{\partial t}(x, 0) = 0, & x\in\Omega, \\ u(x, 0) = 0, & x\in\Omega, \\ u(x, t) = 0, & x\in\partial\Omega,\ t\in[0, T], \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n with a smooth boundary,

$$u(x, t) = (u_1(x, t), u_2(x, t)\cdots u_L(x, t))^T,\ [\nabla_x, \mathscr{A}(x, u)\nabla_x u]$$

denotes an L-tuple whose l-th component is $\sum_{k=1}^{L}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{l,j,k,i}(x, u)\frac{\partial u_k}{\partial x_i}\right)$, $B(x, u)\nabla_x u$ also denotes an L-tuple whose l-th component is $\sum_{k=1}^{L}\sum_{i=1}^{n} b_{l,k,i}(x, u)\frac{\partial u_k}{\partial x_i}$, and $f(x, u) = (f_1(x, u), f_2(x, u), \cdots, f_L(x, u))^T$.

We now make several assumptions which will be referred to as condition (A).

Condition (A) (i) For $(x, p)\in\Omega\times R^L$, the matrix $\mathscr{A}(x, p) = (a_{l,j,k,i}(x, p))$ is symmetric and uniformly positive definite. Any $a_{l,j,k,i}(x, p)$, $\frac{\partial a_{l,j,k,i}(x, p)}{\partial p}$, $\frac{\partial^2 a_{l,j,k,i}(x, p)}{\partial p^2}$, $\frac{\partial^3 a_{l,j,k,i}(x, p)}{\partial p^3}$ are locally bounded.

(ii) For the matrix $B(x, p) = (b_{l,k,i}(x, p))$, any $b_{l,k,i}(x, p)\in C^1$ for $(x, p)\in\Omega\times R^L$ and $\frac{\partial b_{l,k,i}(x, p)}{\partial p}$ are locally bounded.

(iii) For any component $f_l(x, p)$, $\frac{\partial f_l(x, p)}{\partial p}$ is locally bounded.

(iv) $u\in L_\infty(0, T; H^{k+1}\cap W^{1,\infty})$ is a unqiue solution to Problem (1).

* Received April 30, 1985.

The matrix $\mathscr{A}=(a_{l,j,k,i})$ is uniformly positive definite in the sense that there exists a $c_0>0$ such that for all $\xi=(\xi_{lj})$

$$c_0\|\xi\|^2\leqslant\sum a_{l,j,k,i}\xi_{lj}\xi_{ki}, \tag{2}$$

where we have used the norm $\|\xi\|=[[\xi,\ \xi]]^{\frac{1}{2}}$.

Let $S_0=\{(l,\ j);\ 1\leqslant l\leqslant L,\ 1\leqslant j\leqslant n\}$. We shall use two "inner products" on the vector. Suppose $S=\{(l,\ j);\ 1\leqslant l\leqslant L,\ j\in S_l\}$; then for $U=(U_{lj})$ and $W=(W_{lj})$, define $[U,\ W]$ to be an L-tuple given by

$$[U,\ W]=([U,\ W]_1,\ [U,\ W]_2\cdots[U,\ W]_L)^T,$$

$$[U,\ W]_l=\sum_{j\in S_l}U_{lj}W_{lj},$$

$$[[U,\ W]]=\sum_{S}U_{lj}W_{lj}.$$

The variational problem corresponding to Problem (1) is

$$\begin{cases}\left\langle\dfrac{\partial^2 u}{\partial t^2},\ v\right\rangle+a(u;\ u,\ v)=\langle B(x,\ u)\nabla_x u,\ v\rangle+\langle f(x,\ u),\ v\rangle,\\ \left\langle\dfrac{\partial u}{\partial t}(0),v\right\rangle=0,\quad t\in[0,\ T],\ \forall v\in H_0^1,\\ \langle u(0),\ v\rangle=0,\qquad \forall v\in H_0^1,\\ u\in H_0^1(\Omega),\end{cases} \tag{3}$$

where $\langle w,\ v\rangle=(\langle w,\ v\rangle_1,\ \langle w,\ v\rangle_2\cdots\langle w,\ v\rangle_L)^T$, $\langle w,\ v\rangle_l=\int_\Omega w_l(x)v_l(x)dx$, $a(w;\ u,\ v)=\int_\Omega[\mathscr{A}(x,\ w(x))\nabla_x u(x),\ \nabla_x v]\,dx$, and the notation $u\in H_0^1(\Omega)$ means that each component is in $H_0^1(\Omega)$.

We shall approximate the solution to (3) by requiring that the approximate function space and the test function space lie in $\mu=\mu_1\times\mu_2\times\cdots\times\mu_L$, where μ_l is a finite-dimensional subspace of H_0^1.

Let $\mu_l=S_{k,j}^{h,0}(\Omega)\subset H^j(\Omega)\cap H_0^1(\Omega)$ which satisfies[2]:

(i) $\mathscr{P}_k(\Omega)\subset S_{k,j}^{h,0}(\Omega)$,

(ii) let $h\in(0,\ 1)$ and for any $U\in H^r(\Omega)\cap H_0^1(\Omega)$, $r\geqslant 0$, and $0\leqslant s\leqslant\min(r,\ j)$ there exists a constant c_1 independent of h and U such that

$$\inf_{\chi\in S_{k,j}^{h,0}}\|U-\chi\|_{H^s(\Omega)}\leqslant c_1h^\sigma\|U\|_{H^r(\Omega)},$$

where $\sigma=\min(k+1-s,\ r-s)$,

(iii) the inverse property:

$$\|\chi\|_{L_\infty(\Omega)}\leqslant K_0h^{-\frac{n}{2}}\|\chi\|_{L_2(\Omega)},\quad \forall\chi\in S_{k,j}^{h,0}(\Omega).$$

We employ a fully discrete approximation to (3) using finite elements in x and differences in t such that the following hold:

(i) The interval $[0,\ T]$ is partitioned into N equal subintervals

$$0=t_0<t_1<\cdots<t_N=T,\quad t_{j+1}-t_j=\Delta t.$$