EXTRAPOLATION OF FINITE ELEMENT APPROXIMATION IN A RECTANGULAR DOMAIN*

CHEN CHUAN-MIAO
(*Xiangtan University, Xiangtan, Hunan, China*)
LIN QUN
(*Institute of Systems Science, Academia Sinica, Beijing, China*)

Abstract

Recently, the Richardson extrapolation for the elliptic Ritz projection with linear triangular elements on a general convex polygonal domain was discussed by Lin and Lu. We go back in this note to the simplest case, i.e. the bilinear rectangular elements on a rectangular domain which is a parallel case of the one-triangle model in the early work of Lin and Liu. We find that the finite element argument for the Richardson extrapolation with an accuracy of $O(h^4)$ needs only the regularity of $H^{4,\infty}$ for the solution u but the finite difference argument for extrapolation with $O(h^{3+\alpha})$ accuracy needs $u \in C^{5+\alpha}(0 < \alpha < 1)$. Moreover, a formula is suggested to guarantee the extrapolation of $O(h^4)$ accuracy at fine gridpoints as well as at coarse gridpoints.

1. Error expansion for rectangular elements

Let S be a square domain with a square mesh T^h of size $h, u \in H_0^1$ the solution of the Poisson equation with zero boundary condition, $S^h \subset H_0^1$ the piecewise bilinear finite element space over T^h, and $u^h \in S^h$ the Ritz projection defined by

$$(\nabla u^h, \nabla v) = (\nabla u, \nabla v), \quad \forall v \in S^h.$$

Let $u^I \in S^h$ be the interpolant of u.

If $u \in H^{4,\infty}$, we have

$$u^h - u^I = h^2 \Phi + O(h^4 |\log h|) \tag{1}$$

where the coefficient Φ is independent of h.

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Proof of (1) . By Lin and Liu [2], for a function E satisfying $E(\alpha) = E(\beta) = 0$, we have

$$\begin{aligned}\int_\alpha^\beta E dx &= -\frac{1}{12}(\beta-\alpha)^2\int_\alpha^\beta \partial_x^2 E dx - \frac{1}{6}\int_\alpha^\beta Q(x)\partial_x^3 E dx \\ &= -\frac{1}{12}(\beta-\alpha)^2\int_\alpha^\beta \partial_x^2 E dx + \frac{1}{24}\int_\alpha^\beta P(x)\partial_x^4 E dx\end{aligned}$$

with

$$Q(x) = (x-\alpha)(x-\beta)(x-\frac{\alpha+\beta}{2}), \quad P(x) = (x-\alpha)^2(x-\beta)^2,$$

and $P' = 4Q$.

Let $K \in T^h$ with sides 1 and 2 coinciding with x-direction. Then, for $v \in S^h$,

$$\begin{aligned}(\nabla(u^h - u^I), \nabla v) &= (\nabla(u-u^I), \nabla v) \\ &= \sum_K \int_K \partial_y(u-u^I)\partial_y v + \sum_K \int_K \partial_x(u-u^I)\partial_x v = \sum_K I(K) + \sum_K II(K), \\ I(K) &= (\int_1 - \int_2)(u-u^I)\partial_y v dx = -\frac{h^2}{12}(\int_1 - \int_2)\partial_x^2 u \partial_y v dx \\ &- \frac{1}{6}(\int_1 - \int_2)Q(x)(\partial_x^3 u \partial_y v + 3\partial_x^2 u \partial_{xy} v) dx = -\frac{h^2}{12}\int_K \partial_y \partial_x^2 u \partial_y v \\ &- \frac{1}{6}\int_K Q(x)(\partial_y\partial_x^3 u \partial_y v + 3\partial_y\partial_x^2 u \partial_{xy} v), \\ \sum I(K) &= \frac{h^2}{12}\int_S \partial_y^2\partial_x^2 u \cdot v - \frac{1}{6}\int_S Q(x)(\partial_y\partial_x^3 u \partial_y v - 3\partial_y^2\partial_x^2 u \partial_x v),\end{aligned}$$

and, similarly,

$$\sum II(K) = \frac{h^2}{12}\int_S \partial_y^2\partial_x^2 u \cdot v - \frac{1}{6}\int_S Q(y)(\partial_x\partial_y^3 u \partial_x v - 3\partial_x^2\partial_y^2 u \partial_y v).$$

Let $G_z^h \in S^h$ be the discrete Green function of G_z and $g_z \in H_0^1 \cap H^2$ the regular Green function defined by Frehse and Rannacher [1]. One has

$$\int_S |G_z^h - G_z| \le ch^2|\log h|,$$

$$\int_S |D(G_z^h - g_z)| \le ch|\log h|,$$

$$\int_S |D^2 g_z| \le c|\log h|.$$