ON THE OPERATOR EQUATION $SX - XT = Q$

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Abstract

In this talk, we discuss the solvability of the operator equation $SX - XT = Q$, where S and $-T$ are generators of some semigroups of operators.

1. Introduction

Throughout my talk, let $\mathcal{X}$ and $\mathcal{Z}$ be two Banach spaces, and $B(\mathcal{X}, \mathcal{Z})$ be the Banach space of all bounded (linear) operators from $\mathcal{X}$ to $\mathcal{Z}$. Let S and $-T$ be the infinitesimal generators of (C_0)-semigroups $\{G(t) : t \geq 0\} \subset B(\mathcal{Z})$ and $\{H(t) : t \geq 0\} \subset B(\mathcal{X})$, respectively. For $Q \in B(\mathcal{X} \mathcal{Z})$, by a solution of

$$SX - XT = Q \tag{E}$$

we mean an operator X in $B(\mathcal{X}, \mathcal{Z})$ which maps the domain $D(T)$ of T into $D(S)$ such that

$$SXu - XTu = Qu$$

holds for all $u \in D(T)$.

Why are we interested in this equation? This goes back to early 1930s, when P.A.M. Dirac, H. Weyl, Von Neumann, and many of their comtemporaries were developing the mathematical theory of quantum mechanics. And they were very much concerned about stability of the spectrum of self-adjoint operators under small perturbations. In the simplest case, let S and P be bounded operators on a Banach space. If there exists a bounded invertible operator W such that

$$S = W^{-1}(S + P)W,$$

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then the spectrum of $S+P$ remains the same as that of S.

We observe that a sufficient condition for the similarity of $S+P$ and the unperturbed S is the simultaneous solvability of the following pair of equations:

$$SX - XS = Q \quad \text{and} \quad Q + PX = -P.$$

In this case, we have $(I+X)S = (S+P)(I+X)$. Thus $S+P$ is similar to S if $I+X$ is invertible. As a result of this observation, the first step towards a perturbation theory would seemed to be a thorough study of the commutator equation $SX - XS = Q$.

More generally, we consider the equation (E) for two (generally unbounded) operators S and T. We define a map Δ from $B(\mathcal{X}, \mathcal{Z})$ into itself which has as its domain $D(\Delta)$ the set of all those X in $B(\mathcal{X}, \mathcal{Z})$ for which $XD(T) \subset D(S)$ and $SX - XT$ is bounded on $D(T)$, and which sends each such X to the closure of $SX - XT$.

The solvability of (E) is equivalent to the question when is $Q \in R(\Delta)$ (the range of Δ)? This is closely related to Kato-Cook-Rosemblum approach to perturbation of self-adjoint operators, and the existence of "Wave Operator"

$$W_{\pm} = \lim_{t \to \pm\infty} e^{itT} e^{-itS}$$

(unitary or partial isometry) which impliments the similarity between $T = S + P$ and S.

2. Historical Remarks

The equation (E) has a long story starting from the space of matrices and ending up in complex Banach algebra.

(1) D.E. Rutherford (1932) studied the equation (E) with S, T and Q in M_n the space of n by n matrices. He showed that if the eigenvalues of S are distinct from those of $-T$, then (E) has a unique solution $X \in M_n$.

(2) E. Heinz; H. O. Cordes (1951, 1955) showed that when S and T are bounded operators on a Hilbert space , and if there exist constants $a > b$ such that $S + S^* \leq b$ and $T + T^* \geq a$ then Δ^{-1} exists as a bounded operator in $B(\mathcal{X})$ with the following representation

$$X = \Delta^{-1}(Q) = -\int_0^{\infty} e^{tS} Q e^{-tT} dt.$$

(3) M. Rosenblum (1956), by using dunford operational calculus, he was able to show that if there exists a Cauchy domain Ω such that $\sigma(T) \subset \Omega$ and $\sigma(S) \cap \overline{\Omega} = \emptyset$, then $X = \Delta^{-1}(Q)$ exists, and is given by

$$X = \frac{1}{2\pi i} \int_{\delta\Omega} (S - z)^{-1} Q (z - T)^{-1} dz$$