THE FINITE ELEMENT METHOD FOR NONLINEAR ELASTICITY*

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Abstract

In this paper we consider the finite element method for nonlinear elasticity in the case when body force is small. The incremental method and the improved incremental method are investigated, their convergence are proved and the error estimates are obtained.

§1. Introduction

In the present paper we discuss the system of nonlinear elasticity :

$$\begin{cases} -\partial_j(\sigma_{ij}+\sigma_{kj}\partial_k u_i) = f_i, & \Omega, \\ u_i = 0, & \partial\Omega, \end{cases} \quad i=1,2,3 \tag{1.1}$$

where $u=(u_1,u_2,u_3,)^T$ is the displacement vector, $f=(f_1,f_2,f_3)^T$ is the exterior body force, and

$$\sigma_{ij}=\lambda E_{kk}(u)\delta_{ij}+2\mu E_{ij}(u)+o(E),$$

while

$$E(u)=\frac{1}{2}(\nabla u^T+\nabla u+\nabla u^T\nabla u). \tag{1.2}$$

We confine ourselves to the case when body force is sufficiently small. Hence we may suppose (see §5 for details) :

$$\sigma_{ij}=\lambda E_{kk}(u)\delta_{ij}+2\mu E_{ij}(u). \tag{1.3}$$

There are some mathematical results about the system (1.1). Especially, Ciarlet and Destuynder [1] proved the existence and uniqueness of solutions for (1.1) when f is sufficiently small. Bernadou, Ciarlet and Hu [2] proved the convergence of

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semi-discrete incremental methods for (1.1) in the case when f is sufficiently small. Ciarlet [3] summed up these results. These theoretical results and the wide application of finite element methods have promoted the investigation of the finite element methods for solving (1.1).

The present paper proves the existence, the uniqueness and the convergence of the finite element solutions and the convergence of the incremental methods for (1.1) under the condition that $\partial\Omega$ has certain smoothness and $f \in (W^{1,p}(\Omega))^3$ is sufficiently small. Error estimates of the incremental methods are also given.

In the present paper we assume that $\Omega \subset R^3$ is a bounded connected open set with $\partial\Omega$ sufficiently smooth and consisting of finite disjoint simple closed surfaces.

Let $\Omega_h \subset \Omega, h > 0$, be regions composed of a finite number of polyhedrons. To each Ω_h there corresponds a regular triangulation (cf. Ciarlet [7]). We assume that Ω_h satisfy $\max\limits_{x \in \partial\Omega_h} \text{dist}\,(x, \partial\Omega) \le kh^2$, where $k > 0$ is a constant which only depends on Ω.

For convenience, let $\mathcal{W}^{1,p}(\Omega)$ denote $(W^{1,p}(\Omega))^3, \mathcal{C}(\Omega)$ denote $(C(\Omega))^3, \mathcal{H}_0^1(\Omega)$ denote $(H_0^1(\Omega))^3$, etc.

Denote

$$V_h = \left\{ u \in \mathcal{C}(\Omega) | u|_{\Omega \setminus \Omega_h} = 0, u|_K \in \mathcal{P}_2(K), \forall K \in \mathcal{J}_h \right\}$$

where $P_2(K)$ is the set of polynomials of second degree defined on K.

In §2 we will review some theoretical results about the system (1.1). The proofs of these results may be found in Chapter 2 of [3]. Some of the results are given in a more generalized form, but the original proofs remain valid if we note the results of Agmon, Douglis and Nirenberg [4].

In §3 we will investigate such properties as the existence, the uniqueness and the error estimates, etc. of the solutions of the corresponding variational problem of (1.1), when the problem is confined to the finite element space V_h .

In §4 we will investigate incremental methods, and their convergence and the error estimates.

In §5 some additional notes on σ_{ij} are given.

§2. Some theoretical results

We write (1.1) in the form of an operator

$$\begin{aligned} &A \ : \ \mathcal{W}^{m+2,p}(\Omega) \cap \mathcal{W}_0^{1,p}(\Omega) \to \mathcal{W}^{m,p}(\Omega), \\ &Au = f, \end{aligned} \tag{2.1}$$

$A'(0)$ denotes the Fréchet derivative of A at 0. Let

$$\begin{aligned} &\varepsilon(u) = \tfrac{1}{2}(\nabla u^T + \nabla u), \\ &\tilde{\sigma}_{ij}(u) = \lambda\varepsilon_{kk}(u)\delta_{ij} + 2\mu\varepsilon_{ij}(u). \end{aligned}$$