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FINITE ELEMENT EIGENVALUE COMPUTATION ON DOMAINS WITH REENTRANT CORNERS USING RICHARDSON EXTRAPOLATION*

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Abstract

In the presence of reentrant corners or changing boundary conditions, standard finite element schemes have only a reduced order of accuracy even at interior nodal points. This "pollution effect" can be completely described in terms of asymptotic expansions of the error with respect to certain fractional powers of the mesh size. Hence, eliminating the leading pollution terms by Richardson extrapolation may locally increase the accuracy of the scheme. It is shown here that this approach also gives improved approximations for eigenvalues and eigenfunctions which are globally defined quantities.

§1. Introduction

On domains with reentrant corners standard finite element schemes usually suffer from a global loss of accuracy caused by the presence of the local corner singularities. Various methods are devised in the literature for suppressing this "pollution effect", mainly by using systematic mesh refinement near the corner points, or by incorporating "singular" shape functions into the scheme. All these procedures require significant overheads and are not always easy to combine with existing standard routines; see [1] for a survey of these methods. As an alternative, it is proposed in [3] to use Richardson extrapolation for eliminating the leading pollution terms in the error. This approach is based on asymptotic error expansions of the form

$$(u-u_h)(x)=\sum_{n=1}^{N}A_n(x)h_n^{2\alpha_n}+R_h(x)h^2|\log(h)|, \tag{1.1}$$

where h_n are local mesh size parameters, and $\alpha_n<1$ are the exponents in the leading "singularities" at the reentrant corners. Such an expansion is established in [3] for the approximation of the 2-dimensional Poisson equation by linear finite elements on certain locally uniform meshes. On the basis of (1.1), a simple extrapolation procedure yields improved approximations $\tilde{u}_h$ to u, satisfying

$$(u-\tilde{u}_h)(x)=R_h(x)h^2|\log(h)|, \quad x\in\Omega. \tag{1.2}$$

The implementation of this method is relatively easy, compared with the use of singular functions or systematic mesh refinements, particularly for three-dimensional problems. A further advantage is that one can work with (piecewise) uniform meshes which, at least in

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the case of constant coefficients, allows for very economical storage techniques. This makes it possible to solve two- and also certain three-dimensional "corner problems" with high accuracy on small computers.

The remainder term R_h in the expansion (1.1) is integrable on Ω, uniformly as $h \to 0$. Hence, the pointwise error estimate (1.2) implies similar ones also for certain globally defined quantities. For example, in approximating torsion moments one obtains

$$\int_\Omega u\,dx - \int_\Omega \tilde{u}_h dx = O(h^2|\log(h)|). \tag{1.3}$$

In the present paper, we shall extend this result to the approximation of eigenvalues and eigenfuncitons. In particular, an expansion of the form

$$\bar{\lambda}_h - \lambda = \sum_{n=1}^{N} C_n h_n^{2\alpha_n} + O(h^2) \tag{1.4}$$

will be derived for appropriate mean values $\bar{\lambda}_H$ of the discrete eigenvalues, which implies that the extrapolated values $\tilde{\lambda}_h$ satisfy

$$\tilde{\lambda}_h - \lambda = O(h^2). \tag{1.5}$$

This approach may largely improve the eigenvalue approximations already on relatively coarse meshes; some numerical results are given at the end of the paper. Consider, for example, the eigenvalue problem of the Laplacian, $-\Delta$, first on the unit square, and then on the unit square with two slits (see Fig. 2 below). Then, for linear finite elements an error in the smallest eigenvalue of less than 1% requires about 225 nodes for the regular domain, but more than 10,000 nodes for the slit domain. In the latter case, one step of h-extrapolation gives the same accuracy already on a mesh with 900 nodes.

In the following, $L_p(\Omega), 1 \le p \le \infty$, and $H^m(\Omega), H_0^m(\Omega), m \in I\!N$, are the usual Lebesgue and Sobolev spaces on Ω. The norm of $L_p(\Omega)$ is denoted by $\|\cdot\|_p$. The notation $\|\cdot\|_{p;loc}$ refers to the L_p-norm on any fixed subdomain $\Omega' \subset \Omega$ having positive distance to all of the corner points of $\partial\Omega$. The L_2-inner product and norm on Ω are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Finally, the symbol c is used for a generic positive constant which may vary with the context, but is always independent of the mesh size parameter and of the particular functions involved.

§2. The Pollution Effect

Let $\Omega \subset I\!R^2$ be a polygonal domain with reentrant corners or slits. As model situations, we consider the first boundary value problem of the Laplacian operator

$$-\Delta u = f, \quad \text{in } \Omega, \quad u = b, \quad \text{on } \partial\Omega, \tag{2.1}$$

and the corresponding eigenvalue problem

$$-\Delta w = \lambda w, \quad \text{in } \Omega, \quad w = 0, \quad \text{on } \partial\Omega. \tag{2.2}$$

The data f and b are assumed to be smooth, say $f \in C^\alpha$ and $b \in C^{2+\alpha}$. Below, we shall largely use the notation of [3]. Let $\hat{Z} \equiv \{z_n, n = 1, \cdots, \hat{N}\}$ be the set of all corner points of