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ON THE PROBLEM OF BEST RATIONAL APPROXIMATION WITH INTERPOLATING CONSTRAINTS (II)* [1)]

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Abstract

This paper is a continuation of [2]. In this part of the conjoint paper, we establish some results about uniqueness/non-uniqueness, the properties for the set of best approximants, strong uniqueness and continuity of the best approximation operator.

§4. Uniqueness

In [1], Wang has obtained the following important result: Let $R \in \mathbf{R}_1(m,n)$ be a best approximant of f in $\mathbf{R}_1(m,n)$. If $r(x) = f(x) - R(x)$ alternates at least $m+n-d(R)-k+2$ times according to $w(x)$, then the best approximant is unique.

Now we illustrate by an example that the best approximant may be unique as well even if the alternation condition above is not true.

Example 4. Take $m = 0$, $n = 1$, $\mathbf{R}_1(m,n)$ is the same as the one in Example 3. Let $f = x^2$. Then $R = 1 \in \mathbf{R}_1(m,n)$ is the unique best approximant of f in $\mathbf{R}_1(m,n)$. The uniqueness can be verified by computing the maximum value of the function

$$\frac{1}{ax+1} - x^2, \ x \in [-1,1], \ a \in (-1,1).$$

However, $R = 1$ does not have *deviation point* except a *neutral deviation point* $x = 0$. If we draw a curve of the discontinuous function $h(x) \equiv sign\ w(x)(f - R)(x)$, we can see that $h(x)$ has the alternating property. This fact indicates the existence of more general conditions under which the uniqueness is guaranteed. In order to find these conditions, we shall in this paper treat *neutral deviation points* as *non-neutral deviation points*.

Now we state a simple lemma.

Lemma 4. *Let* $f \in \mathbf{C}[a,b] \setminus \mathbf{R}_1(m,n)$, $R \in \mathbf{R}_1(m,n)$, $r(x) = f(x) - R(x)$. *Then for any* $y \in Y(R)$,

$$\liminf_{x \to y+} \frac{|r(x) - r(y)|}{|x-y|^\alpha} = 0, \ \alpha > o \ , \tag{4.1}$$

if and only if there do not exist $\lambda > 0$, $\eta > 0$, *such that*

$$0 < x - y < \eta,$$

$$|r(x)| + \lambda |x-y|^\alpha \le |r(y)|.$$

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For the left limit, a similar conclusion holds.

Let

$$X_1^{\pm}(R) = X_1(R) = \{x : x \in Y(R) \setminus X\},$$

$$X_2^{+}(R) = \{x \in Y(R) : \exists j \ s.t. \ x = x_j \in X, \ k_j \ is \ an \ odd \ integer, \quad \text{and} \ \liminf_{t \to x+} \frac{|r(t) - r(x)|}{(t-x)^{k_j+1}} = 0 \},$$

$$X_3^{+}(R) = \{x \in Y(R) : \ \exists j \ s.t. \ x = x_j \in X, \ k_j \ is \ an \ even \ integer, \quad \text{and} \ \liminf_{t \to x+} \frac{|r(t) - r(x)|}{(t-x)^{k_j+2}} = 0 \},$$

$$X_4^{+}(R) = \{x \in Y(R) \setminus X_3^{+}(R) : \ \exists j \ s.t. \ x = x_j \in X, \ k_j \ is \ an \ even \ integer, \quad \text{and} \ \liminf_{t \to x+} \frac{|r(t) - r(x)|}{(t-x)^{k_j+1}} = 0 \},$$

Similarly, $X_2^-(R), X_3^-(R)$ and $X_4^-(R)$ can be defined. For convenience of discussion, we assume next

$$X_i(R) \equiv X_i^+(R) = X_i^-(R), \quad i = 2, 3, 4.$$

We shall see that the points in $\bigcup_{i=1}^{3} X_i^{\pm}(R)$ play the part of *deviation points* as ordinary best rational approximations.

Let $h(x) = sign\ w(x)r(x)$, $a \le t_1 \le t_2 \le \cdots \le t_p \le b$. If

(i) $t_i \in \bigcup_{j=1}^{3} X_j(R)$ for $i = 1, 2, ..., p$,

(ii) there exist u_s in any neighborhood of t_s such that

$$u_1 < u_2 < ... < u_p,$$

$$h(u_s)h(u_{s+1}) < 0, \quad for \ [t_s, t_{s+1}] \cap X_4(R) = \phi,$$

(iii) for any $\{\hat{t}_i\}_1^k$ that satisfies (i) and (ii), $k \le p$, then we say $r(t)$ is weakly alternate according to $w(t)$ on $\{t_i\}_1^p$, and $\{t_i\}_1^p$ is called a *weak alternant*.

Since u_s, which satisfies condition (ii), can be close to t_s sufficiently, $sign\ h(u_s) = 1(\text{or } -1)$ is a constant. We denote it by $\alpha(t_s)$.

For $i = 1, ..., p-1$, let

$$l_i(R) = \begin{cases} 1, \text{ if } \ (-1)^{c_i(R)} h(t_i-)h(t_{i+1}+) < 0 \text{ and } t_i < t_{i+1}, \\ \quad c_i(R) \ne 0 \text{ (or } c_i(R) = 0, \text{ but } (t_i, t_{i+1}) \cap X_3(R) = \phi), \\ 0, \ \text{otherwise}, \end{cases}$$

where $c_i(R) = card\{[t_i, t_{i+1}] \cap X_4(R)\}$, $for \ t_i < t_{i+1}$.

It is easy to show that $l_i(R)$, $c_i(R)$ depend on R but are independent of t_i $(i = 1, 2, ..., p)$.

Now we can establish the unicity theorem.

Theorem 5. *Let $R \in \mathbf{R}_1(m,n)$ be a best approximation to $f \in C[a,b] \setminus \mathbf{R}_1(m,n)$ from $\mathbf{R}_1(m,n)$. Then the best approximation of f is unique if and only if*

$$m + n + 1 - d(R) - k \le c + \sum_{i=1}^{p-1} l_i(R) \ ,$$