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ON THE PROBLEM OF BEST RATIONAL APPROXIMATION WITH INTERPOLATING CONSTRAINTS (I)* 1)

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Abstract

The aim of this conjoint paper is to discuss the problem of best rational approximation with interpolating constraints. In Part I ,we give the necessary and sufficient conditions for the existence of the best rational approximations and establish some characterizatoin theorems for such approximations. The problems of uniqueness, the properties for the set of best approximations, strong uniqueness and continuity of the best approximation operator are considered in Part II. The results obtained in this paper are the completion and extension of those given in [1].

§1. Introduction

Let m, n and t be nonnegative integers (i.e. $m, n, t \in \mathbf{N}$),

$$a \le x_0 < x_1 < \cdots < x_s \le b,$$

and $\{k_0, k_1, \cdots, k_s\} \subset \mathbf{N}$ such that

$$0 \le k_i \le t,\ i = 0, 1, \cdots, s;\ k \equiv \sum_{i=0}^{s}(k_i + 1) \le m + n.$$

Let

$$\mathbf{H}_i = \{P : P(x) = \sum_{j=0}^{i} a_j x^j,\ a_j \in R\},$$

$$\mathbf{R}(m, n) = \{P/Q : P \in \mathbf{H}_m, Q \in \mathbf{H}_n, P/Q \textit{ irreducible}\}.$$

For $g \in \mathbf{C}^t[a, b]$, we define the set $\mathbf{R}_1(m, n)$ to be

$$\mathbf{R}_1(m, n) = \{R \in \mathbf{R}(m, n) : (R - g)^{(j)}(x_i) = 0,\ i = 0, 1, ..., s;\ j = 0, 1, ..., k_i\}.$$

The problem that we wish to consider in this paper is: For a given $f \in \mathbf{C}[a, b]$, find an element R from $\mathbf{R}_1(m, n)$, such that

$$\|f - R\| = \inf_{T \in \mathbf{R}_1(m,n)} \|f - T\| \equiv d_1(f) \tag{1.1}$$

where $\|\cdot\|$ denotes the Chebyshev norm in $\mathbf{C}[a, b]$.

The set $\mathbf{R}_1(m, n)$ is called the interpolating restricted range. The cardinalities about this set are discussed in [2]. We assume in this paper that $\mathbf{R}_1(m, n) \cap \mathbf{C}[a, b] \ne \phi$ and that the denominators of the elements in $\mathbf{R}_1(m, n) \cap \mathbf{C}[a, b]$ are positive on $[a, b]$.

Next we study the solvability of problem (1.1) and the characterizations of its solutions.

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§2. Existence

For the given $g \in \mathbf{C}^t[a,b]$, we introduce first the following sets:

$$\mathbf{L}(m,n) = \{(P,Q) \in \mathbf{H}_m \times \mathbf{H}_n : (P-Qg)^{(j)}(x_i) = 0, \quad i = 0,1,...,s;\ j = 0,1,...,k_i\}$$

$$\mathbf{R}_0(m,n) = \{P/Q \in \mathbf{R}(m,n) : \exists q \ \ s.t.\ (qP,qQ) \in \mathbf{L}(m,n)\}.$$

Then we introduce the following problem which is related to the problem (1.1): Find $R \in \mathbf{R}_0(m,n)$ such that

$$\|f-R\| = \inf\{\|f-T\| : T \in \mathbf{R}_0(m,n)\} \equiv d_0(f). \tag{2.1}$$

It is easy to deduce that $\mathbf{R}_1(m,n) \subset \mathbf{R}_0(m,n)$, therefore

$$d_0(f) \le d_1(f). \tag{2.2}$$

Next we discuss the solvabilities of problems (1.1) and (2.1) and the relationship between them.

Example 1. Let $m=0$, $n=1$, $[a,b]=[0,1]$, $s=0$, $k_0=0$, $x_0=0$, and $g=1$. Then we have

$$\mathbf{R}_1(m,n) = \{\frac{1}{ax+1} : a \in R\}, \quad \mathbf{R}_0(m,n) = \mathbf{R}_1(m,n) \cup \{0\}.$$

a) Take $f = -2x+1$, then (1.1) is unsolvable. The reason for this is, for any $R \in \mathbf{R}_1(m,n)$, $\|f-R\| > 1$, and $R_k = \dfrac{1/k}{x+1/k} \in \mathbf{R}_1(m,n)$ such that $\|f-R_k\| \longrightarrow 1$. Therefore $d_1(f)=1$. However , problem (2.1) is solvable and $R=0$ is one of its solutions.

b) Take $f=C$ (constant), then for any $a \ge 0$, $R = 1/(ax+1)$ is a solution of problem (1.1) if $C \le 1/2$, while $R=0$ is the unique solution of (2.1) if $C < 1/2$.

c) Take $f = C, C > 1/2$ and $C \ne 1$, then the two problems have the same, and infinitely many, solutions.

This example illustrates that the solvability of (1.1) is a complicated problem compared with the problem of ordinary best rational approximation. For problem (2.1), however, we have the following determinate conclusion.

Theorem 1. *Problem* (2.1) *is solvable for any given* $f \in \mathbf{C}[a,b]$.

Proof. Similarly to the proof given in, [3, p.155], there exists a sequence

$$\{R_k = P_k/Q_k, k = 1,2,\cdots\} \subset \mathbf{R}_0(m,n)$$

such that

$$P_k \longrightarrow \hat{P},\ Q_k \longrightarrow \hat{Q},\ \|Q_k\| = \|\hat{Q}\| = 1 \quad \|P/Q - f\| = d_0(f), \tag{2.3}$$

where P/Q is the irreducible form of $\hat{P}/\hat{Q}$. Now we shall show that $P/Q \in \mathbf{R}_0(m,n)$.

By the definition of $\mathbf{R}_0(m,n)$, there exist q_k such that

$$(\hat{P}_k, \hat{Q}_k) \equiv q_k(P_k, Q_k) \in \mathbf{L}(m,n),$$

and $w(x) \equiv \prod_{i=0}^{s}(x-x_i)^{k_i+1}$ can be divided exactly by q_k. Since q_k's are bounded above , we may assume $q_k \longrightarrow q$ without loss of generality. Then we have from (2.3) that

$$\hat{P}_k \longrightarrow q\hat{P},\ \hat{Q}_k \longrightarrow q\hat{Q}.$$