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A NEW CLASS OF VARIATIONAL FORMULATIONS FOR THE COUPLING OF FINITE AND BOUNDARY ELEMENT METHODS *1)

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§1. Introduction

The coupling of finite element (FEM) and boundary element (BEM) methods has been widely used in the engineering [1, 2]. The coupling method preserves the good features of both FEM, such as applicability to nonhomogeneous and nonlinear problems, and BEM, such as applicability to problems on unbounded domains. At present, there are two different concepts for this coupling. One consists of subdividing the original domain $\Omega \subset R^n$ into a finite number of subregions and using either FEM or BEM in each of them, where the latter lies on all boundaries of the subregions. The second concept, which we will not consider, uses BEM for the modelling of special finite element functions; see Schnack [3]. Error estimates for a coupling of FEM-BEM have been presented by Johnson and Nedelec [4] for a special exterior boundary value problem with the Laplacian. Wendland [5] extended their approach to more general equations and problems but needed a rather strong assumption, namely a compact perturbation of the identity, for the coupling operator between FEM and BEM. Unfortunately, this assumption is violated for many practical problems in e.g. elasticity. In this case, Wendland required a more careful coupling of the elements instead by imposing higher regularity of the finite elements at the coupling boundary and requiring a faster mesh refinement of the boundary elements than of the finite elements. In all the cases cited, the error estimates hold only for $0 < h_1,\ h_2 \le h_0$, where h_1, h_2 denote the parameters of mesh width corresponding to FEM and BEM, and $h_0 > 0$ is an unknown constant. Hence in the practical computation, it is very hard to determine how small h_1 and h_2 must be to ensure that the error estimates hold.

In this paper, we present a new formulation for coupling of FEM and BEM, which preserves the coercive property of the original problem. Therefore, the strong assumption for the coupling operator required by Wendland will no longer be needed. Furthermore, the error estimates hold for all $h_1 > 0$ and $h_2 > 0$.

§2. The New Variational Formulations

Let Ω^c be the complement of a bounded regular domain in R^2 with boundary Γ. We consider the following two problems as examples.

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2.1. Dirichlet problem for Poisson's equation

Consider the boundary value problem:

$$\begin{cases} -\Delta u = f, \text{ in } \Omega^c, \\ u = 0, \text{ on } \Gamma, \\ u \text{ is bounded, when } |x| \to +\infty \end{cases} \tag{2.1}$$

where f has its support in a bounded subdomain Ω_1 of Ω^c. Let Γ_2 be the boundary of Ω_1 in Ω^c and Ω_2 be $\Omega^c \setminus \bar{\Omega}_1$ (see Fig. 1). We solve the problem (2.1) by using the coupling of FEM and BEM. Consider the equivalent system of equations:

$$\begin{cases} -\Delta u_1 = f, & \text{in } \Omega_1, \\ -\Delta u_2 = 0, & \text{in } \Omega_2, \\ u_1 = u_2, & \text{on } \Gamma_2, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \lambda, & \text{on } \Gamma_2 \\ u_1 = 0, & \text{on } \Gamma, \\ u_2 \text{ is bounded, when } |x| \to \infty \end{cases} \tag{2.2}$$

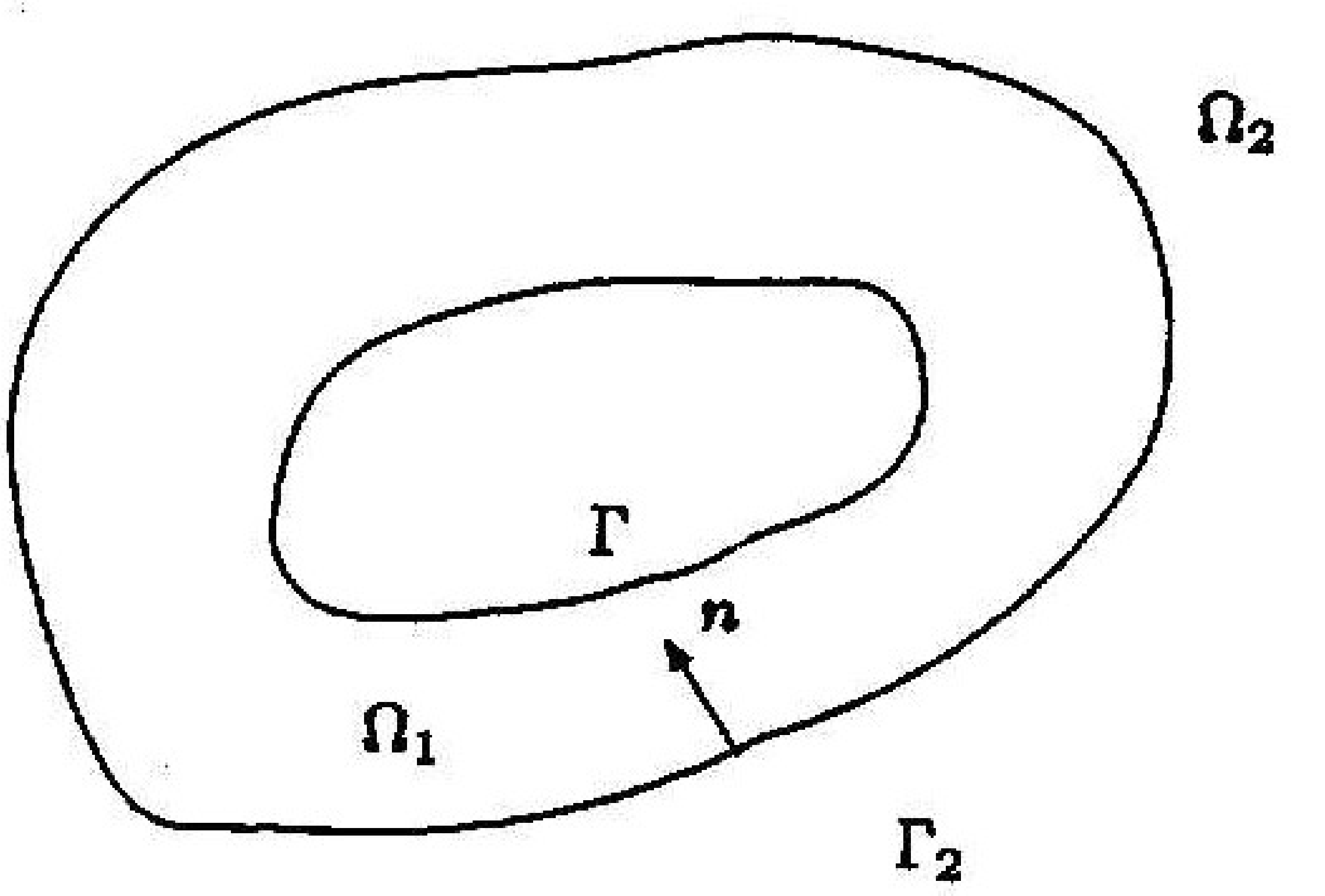

Fig. 1

where $u_i = u|_{\Omega_i}, i = 1, 2$, and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative to $\Gamma_2 = \partial\Omega_2$ (see FIg. 1). Since $-\Delta u = 0$ in Ω_2, using Green's formula we obtain

$$u(x) = \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u(y) ds_y - \int_{\Gamma_2} G(x,y)\lambda(y) ds_y + \alpha, \quad \forall x \in \Omega_2, \tag{2.3}$$

where $G(x,y) = \dfrac{1}{2\pi} \log|x-y|, x \neq y; n_y$ denotes the outward unit normal to $\Gamma_2 = \partial\Omega_2$ at $y \in \Gamma_2$, and α is a constant. By the properties of the single-layer and double-layer potentials, we obtain the following relationship between λ and $u|_{\Gamma_2}$:

$$\frac{1}{2}u(x) = \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_y} u(y) ds_y - \int_{\Gamma_2} G(x,y)\lambda(y) ds_y + \alpha \quad \forall x \in \Gamma_2. \tag{2.4}$$

Furthermore, using properties of the derivatives of the single-layer and double-layer potentials [6, 7], we get

$$\frac{1}{2}\lambda = \int_{\Gamma_2} \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y) ds_y - \int_{\Gamma_2} \frac{\partial G(x,y)}{\partial n_x} \lambda(y) ds_y, \quad \forall x \in \Gamma_2, \tag{2.5}$$

where

$$\int_{\Gamma_2} \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y) ds_y = \frac{d}{ds_x} \int_{\Gamma_2} G(x,y) \frac{du(y)}{ds_y} ds_y, \quad \forall x \in \Gamma_2.$$