Journal of Computational Mathematics, Vol.8, No.2, 1990, 135-146.

BIVARIATE POLYNOMIAL NATURAL SPLINE INTERPOLATION TO SCATTERED DATA *

Li Yue-sheng Guan Lü-tai
(*Department of Computer Science, Zhongshan University, Guangzhou, China*)

Abstract

By means of the theory of spline interpolation in Hilbert spaces, the bivariate polynomial natural spline interpolation to scattered data is constructed. The method can easily be carried out on a computer, and parallelly generalized to high dimensional cases as well. The results can be used for numerical integration in higher dimensions and numerical solution of partial differential equations, and so on.

There are many papers about multivariate spline interpolation and a comprehensive and important survey is given almost every two years in the world [1]-[5]. Now, B-splines, B-nets, thin plate splines and radial functions[6] are useful tools for interpolation, but a method which can easily be carried out on a computer with better variational properties is not found.

The first author of this paper has pointed out that based on variational consideration with multiple restrictions via a generalized Euler equaiton, a simple, convenient and practical method for multivariate spline interpolation could be obtained, and in fact a suitable generalized blending spline function space for solving problems of multivariate optimal interpolation to scattered data throughout a rectangle with continuous boundary conditions and discrete boundary conditions has been constructed in [7], [8]. But it needs to be revised and has not yet been published in magazines.

By means of the theory and methods of spline interpolation in Hilbert spaces, we treat again the bivariate polynomial natural spline interpolation to scattered data proposed in [8]. The method is simple and convenient; the solution has better variational properties and suitable smoothing properties. It suits the solution of the problem of bivariate interpolation to scattered data without boundary conditions and can be generalized to similar multivariate interpolation problems. The results can be used for numerical integration in higher dimensions, the computer solution of partial differential equations and so on.

The notations in this paper can be found in [9], [10].

§1. Selection of Spaces and Definition of Operators

Let $X = H^{m,n}(R) = \left\{u(x,y)\Big|\frac{\partial^{m+n}u}{\partial x^m\partial y^n} \in L_2(R), \frac{\partial^{\alpha+\beta}u}{\partial x^\alpha \partial y^\beta}\right.$ is a absolutely continuous function, $\left.\alpha=\overline{0,m-1}, \beta=\overline{0,n-1}; (x,y)\in R=[a,b]\times[c,d]\right\}$.

Let

$$Y = L_2(R) \times \prod_{\nu=0}^{n-1} L_2[a,b] \times \prod_{\mu=0}^{m-1} L_2[c,d]$$

* Received September 4, 1987.

be a product space where $\prod_{\nu=0}^{n-1} L_2[a,b]$ is a product space constructed by n Hilbert space $L_2[a,b]$, and $Z = R^p$ be a p-dimensional Euclidean space. Suppose that they are all Hilbert spaces.

We define a continuous linear operator $T : X \to Y$ with

$$T = t_0 \times \prod_{\nu=0}^{n-1} t_1^{(\nu)} \times \prod_{\mu=0}^{m-1} t_2^{(\mu)}$$

being a product operator mapping X onto product space Y, where

$$t_0 : X \to L_2(R), t_0(u) = u^{(m,n)}(x,y) = \frac{\partial^{m+n} u(x,y)}{\partial x^m \partial y^n},$$

$$t_1^{(\nu)} : X \to L_2[a,b], t_1^{(\nu)}(u) = u^{(m,\nu)}(x,d) = \left.\frac{\partial^{m+\nu} u(x,y)}{\partial x^m \partial y^\nu}\right|_{y=d},$$

$$\nu = 0, \cdots, n-1,$$

$$t_2^{(\mu)} : X \to L_2[c,d], t_2^{(\mu)}(u) = u^{(\mu,n)}(b,y) = \left.\frac{\partial^{\mu+n} u(x,y)}{\partial x^\mu \partial y^n}\right|_{x=b}$$

$$\mu = 0, \cdots, m-1,$$

and define a linear continuous operator $A : X \to Z$ with

$$Au = [u^{(\alpha,\beta)}(x_i, y_i)], \quad \alpha \in I_i, \beta \in J_i, i = \overline{1,N}$$

with $u(x,y) \in X$, the scattered interpolation nodes $(x_i, y_i) \in R$ and $I_i \subset I = \{0, \cdots, m-1\}$, $J_i \subset J = \{0, \cdots, n-1\}$ being arbitrary index sets. The total number of indices is denoted by p.

Given p real numbers $z_i^{\alpha\beta}(\alpha \in I_i, \beta \in J_i, i = \overline{1,N})$, we consider the following spline interpolation problem in Hilbert spaces, which is called bivariate polynomial natural spline interpolation problem, and its solution as bivariate polynomial natural spline interpolant.

Problem P. Find a function $\sigma(x,y) \in X$ satisfying

$$\|T\sigma\|_Y^2 = \min_{x \in I_z} \|Tx\|_Y^2$$

where

$$I_z = \{u \in X | Au = z, z = [z_i^{\alpha\beta}]_{i=1,\alpha\in I_i,\beta\in J_i}^N\}$$

and

$$\|Tx\|_Y^2 = \langle Tx, Tx\rangle_Y = \langle t_0 x, t_0 x\rangle_{L_2(R)} + \sum_{\nu=0}^{n-1} \langle t_1^{(\nu)} x, t_1^{(\nu)} x\rangle_{L_2[a,b]} + \sum_{\mu=0}^{m-1} \langle t_2^{(\mu)} x, t_2^{(\mu)} x\rangle_{L_2[c,d]}$$

$$= \iint_R (u^{(m,n)}(x,y))^2 dxdy + \sum_{\nu=0}^{n-1} \int_a^b (u^{(m,\nu)}(x,d))^2 dx + \sum_{\mu=0}^{m-1} \int_c^d (u^{(\mu,n)}(b,y))^2 dy.$$