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A TRILAYER DIFFERENCE SCHEME FOR ONE-DIMENSIONAL PARABOLIC SYSTEMS*

Chen Guang-nan
(*Institute of Applied Physics and Computational Mathematics, Beijing, China*)

In order to obtain the numerical solution for a one-dimensional parabolic system, an unconditionally stable difference method is investigated in [1]. If the number of unknown functions is M, for each time step only M times of calculation are needed. The rate of convergence is $O(\tau + h^2)$. On the basis of [1], an alternating calculation difference scheme is presented in [2]; the rate of convergence is $O(\tau^2 + h^2)$. The difference schemes in [1] and [2] are economic ones. For the α-th equation, only U_α is an unknown function; the others, $U_\beta(\beta = 1, 2, \cdots, \alpha-1, \alpha+1, \cdots, M)$, are given evaluated either in the last step or in the present step. So the practical calculation is quite convenient.

The purpose of this paper is to derive a trilayer difference scheme for one-dimensional parabolic systems. It is shown that the scheme is also unconditionally stable and the rate of convergence is $O(\tau^2 + h^2)$.

§1

On the domain $D\{0 < x < 1, 0 < t \le T\}$, we consider the partial differential equations

$$\frac{\partial}{\partial t}u_\alpha(x,t) = \sum_{\beta=1}^{M} \frac{\partial}{\partial x}\Big[K_{\alpha\beta}(x,t)\frac{\partial}{\partial x}u_\beta(x,t)\Big], \alpha = 1, \cdots, M, \tag{1}$$

with the initial and boundary conditions

$$u_\alpha(x,0) = u_\alpha^0(x), u_\alpha(0,t) = 0, u_\alpha(1,t) = 0, \alpha = 1, \cdots, M. \tag{2}$$

Suppose that the coefficients of equations (1) satisfy the following conditions:
K1°. $K_{\alpha\beta}(x,t) = K_{\beta\alpha}(x,t)$;
K2°. there exist positive constants $\sigma_1 > 0, \sigma_2 > 0$, such that

$$\sigma_1 \sum_{\alpha=1}^{M} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{M} K_{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \le \sigma_2 \sum_{\alpha=1}^{M} \xi_\alpha^2, \quad (x,t) \in D,$$

for any M-dimensional real vectors $\vec{\xi} \in I\!R^m$;

K3°. the coefficients $K_{\alpha\beta}$ of equations (1) are sufficiently smooth on the domain $D\{0 \le x \le 1, 0 \le t \le T\}$ and especially, there exists a constant $K > 0$, so that $|K_{\alpha\beta}(x,t)| < K, \left|\frac{K_{\alpha\beta}(x,t+\tau) - K_{\alpha\beta}(x,t)}{\tau}\right| < K$.

Since the coefficients satisfy condition K2°, equations (1) belong to the parabolic system. In addition, we assume that there exist unique sufficiently smooth solutions of equations (1) with initial and boundary conditions (2).

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We solve the problem (1)–(2) by the difference method. Divide the intervals $[0, 1]$ and $[0, T]$ into J and N points respectively. The space step is $h = 1/J$ and the time step is $\tau = T/N$. Let $\omega_h = \{x_j = jh | j = 0, 1, \cdots, J\}$ and $\omega_\tau = \{t^n = n\tau | n = 0, 1, \cdots, N\}$. The set of all net points on the domain $\bar{D}$ is denoted by $\bar{\Omega} = \omega_h \times \omega_\tau$, and $\Omega = \bar{\Omega} \cap D$.

Let $U(x, t)$ and $V(x, t)$ be the discrete functions, defined on the set $\bar{\Omega}$. Introduce the following notations:

$$U^n \equiv U_j^n = U(jh, n\tau),$$

$$\begin{aligned} U_{\bar{x}}^n &= U_{\bar{x},j}^n = \frac{1}{h}(U_j^n - U_{j-1}^n), \quad U_x^n = U_{x,j}^n = \frac{1}{h}(U_{j+1}^n - U_j^n), \\ U_{\bar{t}}^n &= U_{\bar{t},j}^n = \frac{1}{\tau}(U_j^n - U_j^{n-1}), \quad U_{\overset{\circ}{t}}^n = U_{\overset{\circ}{t},j}^n = \frac{1}{2\tau}(U_j^{n+1} - U_j^{n-1}). \end{aligned} \tag{3}$$

Define the following scalar products and norms:

$$(U^n, V^n) = \sum_{j=1}^{J-1} U_j^n V_j^n h, \qquad (U^n, V^n] = \sum_{j=1}^{J} U_j^n V_j^n h,$$
$$\|U^n\| = \sqrt{(U^n, U^n)}, \quad \|U^n]| = \sqrt{(U^n, U^n]}, \quad \|U^n\|_\infty = \max_{x \in \omega_h} |U(x, t^n)|.$$

If $U_0^n = U_J^n = 0$ in the interval $0 \le x \le 1$, then there is Green's difference formula

$$(U^n, V_x^n) = -(U_{\bar{x}}^n, V^n] \tag{4}$$

and the relations [3]

$$\|U^n\| \le \|U^n\|_\infty \le \frac{1}{2}\|U_{\bar{x}}^n]|. \tag{5}$$

For the problem (1)–(2), finite difference equations may be constructed in various ways. If we use explicit difference schemes, we have to consider the restriction of the stability condition and require small computational steps. If we adopt fully implicit difference schemes, the iterative computation leads to a huge amount of calculation; so we must consider economic schemes, which both are unconditionally stable and require small amount of calculation. For instance, the following difference scheme is investigated in [1]:

$$\begin{gathered} U_{\alpha,\bar{t}}^{n+1} = \sum_{\beta=1}^{\alpha-1} \left(a_{\alpha\beta}^{n+1} U_{\beta,\bar{x}}^{n+1}\right)_x + \left(\theta a_{\alpha\alpha}^{n+1} U_{\alpha,\bar{x}}^{n+1} + (1-\theta) a_{\alpha\alpha}^n U_{\alpha,\bar{x}}^n\right)_x + \sum_{\beta=\alpha+1}^{M} \left(a_{\alpha\beta}^n U_{\beta\bar{x}}^n\right)_x, \\ (x, t) \in \Omega, \alpha = 1, \cdots, M. \end{gathered} \tag{6}$$

where $0.5 \le \theta \le 1$ is an arbitrarily selected parameter, and

$$a_{\alpha\beta}^n = a_{\alpha\beta,j}^n = K_{\alpha\beta}\big((j - \frac{1}{2})h, n\tau\big), \quad \alpha, \beta = 1, \cdots, M.$$

In [2], an alternating calculation difference scheme is considered:

$$\begin{gathered} U_{\alpha,\bar{t}}^{2n+1} = \sum_{\beta=1}^{\alpha-1} (a_{\alpha\beta}^{2n+1} U_{\beta,\bar{x}}^{2n+1})_x + \frac{1}{2}(a_{\alpha\alpha}^{2n+1} U_{\alpha,\bar{x}}^{2n+1} + a_{\alpha\alpha}^{2n} U_{\alpha,\bar{x}}^{2n})_x + \sum_{\beta=\alpha+1}^{M} (a_{\alpha\beta}^{2n} U_{\beta,\bar{x}}^{2n})_x, \\ (x, t) \in \Omega, \alpha = 1, \cdots, M, \end{gathered} \tag{7}$$