Journal of Computational Mathematics, Vol.9, No.2, 1991, 115–124.

POISSON DIFFERENCE SCHEMES FOR HAMILTONIAN SYSTEMS ON POISSON MANIFOLDS

Wang Dao-liu
(*Computing Center, Chinese Academy of Sciences, Beijing, China*)

Abstract

In this paper a systematical method for the construction of Poisson difference schemes with arbitrary order of accuracy for Hamiltonian systems on Poisson manifolds is considered. The transition of such difference schemes from one time-step to the next is a Poisson map. In addition, these schemes preserve all Casimir functions and, under certain conditions, quadratic first integrals of the original Hamiltonian systems. Especially, the arbitrary order centered schemes preserve all Casimir functions and all quadratic first integrals of the original Hamiltonian systems.

§1. Introduction

For the Hamiltonian system

$$\frac{dz}{dt} = J^{-1}\nabla H(z), \qquad z \in \mathbf{R}^{2n}$$

where $H(z) \in \mathbf{C}^\infty(\mathbf{R}^{2n})$ is a Hamiltonian function, Feng Kang et al. have developed a general method for the construction of symplectic difference schemes via generating functions [3–6]. Such difference schemes preserve the symplecticity of the phase flow and the quadratic first integrals of the original Hamiltonian systems. The method used in [5] can be generalized to the following Hamiltonian system

$$\frac{dz}{dt} = K\nabla H(z), \qquad z \in \mathbf{R}^n \tag{1}$$

where $H(z) \in \mathbf{C}^\infty(\mathbf{R}^n)$, K is an anti-symmetric scale matrix, maybe singular. This system could appear such as in the semi-discretization of infinite dimensional Hamiltonian systems in space variables. The case of non-singular K has been considered in [12]. At that case n is even and a Poisson map coincides with a symplectic map. For a given generator map, a Poisson (or symplectic) map uniquely determines a gradient map, further a scalar function (up to a constant), and vice versa. So the scalar function is called a generating function. Here a given Poisson map can also determine a scalar function. This scalar function, in another way, can also determine a Poisson map. But we do not know if it is the original one. This is the present difficulty. Fortunately, for the phase flow of the system (1), we

* Received January 17, 1988.
1) The Project supported by National Natural Science Foundation of China.

can directly give its time-dependent generating function which satisfies a Hamilton Jacobi equation. When this result is obtained, the remainder is similar to [5].

In sec. 2 we consider some properties of the Hamiltonian system (1). Its phase flow is a one parameter group of Poisson maps. In sec. 3, it is shown that for the phase flow of the system (1) there exists generating functions and its generating functions satisfy some Hamilton Jacobi equation. This generating function can be expressed as a power series for analytic $H(z)$, whose coefficients can be recursively determined. Truncate it then Poisson difference schemes approximating to (1) with arbitrary order of accuracy are obtained (in sec. 4). Sec. 5 is about conservation laws. All Casimir functions and quadratic first integrals, under some conditions, are preserved. Especially, the arbitrary order centered schemes preserve all Casimir functions and quadratic first integrals of the original Hamiltonian systems.

We shall limit ourselves to the local case throughout the paper.

§2. Poisson Manifolds and Hamiltonian Systems

Let $\mathbf{R}^n$ be an n-dim real space. The elements of $\mathbf{R}^n$ are n-dim column vectors $z = (z_1, \cdots, z_n)^T$. The superscript T represents the matrix transpose.

Let $\mathbf{C}^\infty(\mathbf{R}^n)$ be the space of smooth real valued functions on $\mathbf{R}^n$. For $H(z) \in \mathbf{C}^\infty(\mathbf{R}^n)$, we denote $\nabla H = \nabla_z H = (H_{z_1}, \cdots, H_{z_n})^T$, $H_z = (H_{z_1}, \cdots, H_{z_n})$ and $H_{z_i} = \partial H/\partial z_i$. So $\nabla H(z) = (H_z(z))^T$.

For a given anti-symmetric $n \times n$ matrix K(maybe singular), we can define a binary operation $\{\cdot,\cdot\}$ on $\mathbf{C}^\infty(\mathbf{R}^n)$ by

$$\{F, H\} = (\nabla F)^T K \nabla H, \qquad \forall F, H \in \mathbf{C}^\infty(\mathbf{R}^n).$$

Evidently, it is bilinear, anti-symmetric and satisfies the Jacobi identity, i.e.,

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0, \qquad \forall F, G, H \in \mathbf{C}^\infty(\mathbf{R}^n).$$

From the definition it follows that it also satisfies Leibniz identity

$$\{FG, H\} = F\{G, H\} + \{F, H\}G.$$

So $\{\cdot,\cdot\}$ is a Poisson bracket on $\mathbf{R}^n$. $\mathbf{R}^n$ equipped with the Poisson bracket is called a Poisson manifold, still denoted by $\mathbf{R}^n$

A map $z \to \widehat{z} = g(z) : \mathbf{R}^n \to \mathbf{R}^n$ is called a Poisson map if it is a (local) diffeomorphism and preserves the Poisson bracket, i.e.,

$$\{F \circ g, H \circ g\} = \{F, H\} \circ g, \qquad \forall F, H \in \mathbf{C}^\infty(\mathbf{R}^n).$$

It is easy to verify that it is equivalent to

$$g_z(z) K (g_z(z))^T = K,$$

i.e., the Jacobian matrix g_z is everywhere Poisson; here a matrix M is called Poisson if

$$MKM^T = K.$$

Definition. *A function $C(z) \in \mathbf{C}^\infty(\mathbf{R}^n)$ is called a Casimir function if*

$$\{C(z), F(z)\} = 0, \qquad \forall F \in \mathbf{C}^\infty(\mathbf{R}^n).$$

Proposition 1. $C(z) \in \mathbf{C}^\infty(\mathbf{R}^n)$ is a Casimir function if and only if

$$K \nabla C(z) = 0, \qquad \forall z \in \mathbf{R}^n.$$