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ON DISCRETE SUPERCONVERGENCE PROPERTIES OF SPLINE COLLOCATION METHODS FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS*1)

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Abstract

It is shown that the error corresponding to certain spline collocation approximations for nonlinear Volterra integral equations of the second kind is the solution of a nonlinearly perturbed linear Volterra integral equation. On the basis of this result it is possible to derive general estimates for the order of convergence of the spline solution at the underlying mesh points. Extensions of these techniques to other types of Volterra equations are indicated.

§1. Introduction

Consider the nonlinear Volterra integral equation of the second kind

$$y(t) = g(t) + \int_0^t k(t,s,y(s))ds, \quad t \in I := [0,T], \tag{1.1}$$

where $g: I \to R$ and $k: S \times R \to R$ (with $S := \{(t,s) : 0 \le s \le t \le T\}$) denote given continuous functions, which are assumed to be such that (1.1) has a unique solution $y \in C(I)$. Suppose that $u: I \to R$ is an approximation to y satisfying

$$\|y-u\|_\infty := \sup\{|y(t)-u(t)| : t \in I\} = \sigma(h^p) \quad p > 0, \tag{1.2}$$

as $h \to 0$. Here, $h = h^{(N)}$ is the diameter of the underlying mesh $\Pi_N:\ 0 = t_0 < t_1 < \cdots < t_N = T$ (with $t_n = t_n^{(N)}$): $h := \max\{t_{n+1} - t_n : 0 \le n \le N-1\}$. Often u converges faster to y on the mesh Π_N than on I, i.e. there exists a $p^* > p$ so that

$$\max\{|y(t)-u(t)| : t \in \Pi_N\} = \sigma(h^{p^*}). \tag{1.3}$$

We then say that u exhibits discrete (or local) superconvergence of order P^* at the mesh points.

This paper is concerned with the following question: assuming that we have established a global convergence result of the form (1.2), how can we verify if the approximation u (obtained, e.g., by collocation in some finite-dimensional function space)

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where

$$A(t,s) := \frac{\partial k(t,s,y)}{\partial y}\bigg|_{y=y(s)}$$

and

$$(Be)(t) := -\frac{1}{2}\int_0^t \frac{\partial^2 k(t,s,y)}{\partial y^2}\bigg|_{y=w(s)} \cdot e^2(s)ds,$$

with $w(s) := y(s) + \theta(s)e(s)$ *for some* $\theta \in (-1,0)$.

Proof. It follows from (1.1) and (2.7) that $e(t)$ satisfies

$$e(t) = r(t) + \int_0^t \{k(t,s,y(s)) - k(t,s,u(s))\}ds, \quad t \in I.$$

The integrand can be written as

$$k(t,s,y(s)) - k(t,s,y(s)-e(s)) = k(t,s,y(s)) - \frac{\partial k(t,s,y)}{\partial y}\bigg|_{y=y(s)} \cdot e(s) + \frac{1}{2}\frac{\partial^2 k(t,s,y)}{\partial y^2}\bigg|_{y=w(s)} \cdot e^2(s),$$

where, by Taylor's formula, $w(s) := y(s) + \theta(s)e(s)$, with $-1 < \theta < 0$. This yields (2.8).

In an analogous way we obtain an expression for $e_{it}(t) : +y(t) - u_{it}(t)$.

Lemma 2.2. *The iterated collocation error* $e_{it}(t)$ *corresponding to the iterated collocation solution* $u_{it}(t)$ *given by* (2.3) *is related to* $e(t)$ *by*

$$e_{it}(t) = \int_0^t A(t,s)e(s)ds + (Be)(t), \quad t \in I, \tag{2.9}$$

with $A(t,s)$ *and* $(Be)(t)$ *as in Lemma* 2.1.

Lemma 2.3. *Let* $R(t,s)$ *be the solution of the resolvent equation*

$$R(t,s) = -A(t,s) + \int_s^t A(t,v)R(v,s)dv, \quad (t,s) \in S, \tag{2.10}$$

where the kernel $A(t,s)$ *has been introduced in Lemma 2.1. Then* $e(t)$ *solves the non-linearly perturbed linear Volterra integral equation (2.8) if, and only if, it satisfies*

$$e(t) = r(t) - \int_0^t R(t,s)r(s)ds + (Be)(t) - \int_0^t R(t,s)(Be)(s)ds, \ t \in I. \tag{2.11}$$

Proof. Setting $F(t) := r(t) + (Be)(t)$ it follows from the classical Volterra theory (compare also [8,pp. 189-193]) that the solution of

$$e(t) = F(t) + \int_0^t A(t,s)e(s)ds, \quad t \in I,$$

is given by

$$e(t) = F(t) + \int_0^t R(t,s)F(s)ds, \quad t \in I,$$

where the resolvent kernel $R(t,s)$ associated with $A(t,s)$ is defined by (2.10). This yields (2.11). Obviously, the above steps are reversible.