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OPTIMAL INTERPOLATION OF SCATTERED DATA ON A CIRCULAR DOMAIN WITH BOUNDARY CONDITIONS *

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Abstract

Optimal interpolation problems of scattered data on a circular domain with two different types of boundary value conditions are studied in this paper. Closed-form optimal solutions, a new type of spline functions defined by partial differential operators, are obtained. This type of new splines is a generalization of the well-known L_g-splines and thin-plate splines. The standard reproducing kernel structure of the optimal solutions is demonstrated. The new idea and technique developed in this paper are finally generalized to solve the same interpolation problems involving a more general class of partial differential operators on a general region.

§1. Introduction

The "thin plate" splines were first introduced by Duchon[5] over an unbounded domain $\Omega = R^2$. A multivariable interpolation problem relating to this type of thin plate splines was also studied by Meingnet[13]. Based on basis functions including thin plate splines, a numerical method for the interpolation problem of scattered data over a finite domain without boundary conditions was presented by Dyn and Levin[6]. A closed-form solution to the interpolation problem of scattered data over a circular domain with boundary conditions, which leads to the so-called "biharmonic spline" (a second order thin plate spline), was given via the variational technique by Li[12]. Other related results may also be found in, for example, Dyn and Wahba [7], Freeden [9, 10], Li [11], Utreras [15, 16], and Wahba [19]. Some interesting applications of the biharmonic spline to state-constrained minimum-energy optimal control problems with steady-state distributed harmonic systems were shown in Chen [1, 2, 3], where explicit closed-form optimal control and state functions were obtained from the reproducing kernel Hilbert space approach.

In this paper, we will study two optimal interpolation problems of scattered data on a circular domain with two different types of boundary value conditions. Closed-form solutions to the two corresponding problems posed below will be obtained. The optimal solutions will be defined as, in a minimum-norm interpolation sense, a new

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type of spline functions defined by certain partial differential operators with boundary value conditions, which is a natural generalization of the well-known L_g and thin plate splines. Moreover, the elegant reproducing kernel structure for these two different types of splines will also be demonstrated. Finally, the idea and technique will be generalized to solve the same interpolation problems involving a more general class of partial differential operators on a general region.

§2. Statement of Problems

Let Ω be the open disk of radius a and centered at the origin in R^2 with a boundary Γ. For a nonnegative integer k, let $H^k = H^k(\Omega)$ be the usual (real) Sobolev space of order k on Ω, endowed with the inner product and norm defined respectively by

$$\langle f, g\rangle_{H^k} = \sum_{\alpha_1+\alpha_2\le k} \int_\Omega \Big(\frac{\partial^{\alpha_1+\alpha_2}}{\partial^{\alpha_1}x\partial^{\alpha_2}y} f\Big)\Big(\frac{\partial^{\alpha_1+\alpha_2}}{\partial^{\alpha_1}x\partial^{\alpha_2}y} g\Big)dxdy$$

and

$$\|\cdot\|_{H^k} = (\langle\cdot,\cdot\rangle_{H^k})^{1/2},$$

where $\alpha_1, \alpha_2 \ge 0$. Denote by $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ the Laplacian operator as usual. Given m real-valued continuous functions $\{\phi_i\}_{i=0}^{m-1}$ defined on Γ, and a set of scattered data $\{z_i\}_{i=1}^{l}$, the main objective of this paper is to solve the following two constrained minimization problems:

$$\underset{w\in H^m(\Omega)}{\text{minimize}} : \int_\Omega (\Delta^m w)^2 dxdy \tag{2.1}$$

subject to

$$w(x_i, y_i) = z_i, \quad (x_i, y_i) \in \Omega, \quad i = 1, \cdots, l \tag{2.2}$$

and either

$$w|_\Gamma = \phi_0, \quad \Delta w|_\Gamma = \phi_1, \cdots, \Delta^{m-1} w|_\Gamma = \phi_{m-1}, \tag{2.3a}$$

or

$$w|_\Gamma = \phi_0, \quad \frac{\partial w}{\partial r}\Big|_\Gamma = \phi_1, \cdots, \frac{\partial^{m-1} w}{\partial r^{m-1}}\Big|_\Gamma = \phi_{m-1}, \tag{2.3b}$$

where $\{(x_i, y_i)\}_{i=1}^{l}$ are l distinct interior points in Ω and $\partial/\partial r$ is the outward normal derivative on Γ. For convenience, we will call the problem with boundary conditions (2.3a) Problem A and the one with (2.3b) Problem B.

§3. Construction of the Optimal Solutions

In this section, we will show that the optimal solutions to Problems A and B have the same form:

$$w^* = u^* + \sum_{k=0}^{m-1} w_k,$$