Journal of Computational Mathematics, Vol.10, No.4, 1992, 321–338.

MINIMAX METHODS FOR OPEN-LOOP EQUILIBRA IN N-PERSON DIFFERENTIAL GAMES PART III: DUALITY AND PENALTY FINITE ELEMENT METHODS*[1)]

Goong Chen
(*Department of Mathematics, Texas A & M University, USA*)
Wendell H. Miies
(*Research and Development Department, Standard Oil Company of Ohio, USA*)
Wan-Hua Shaw
(*Department of Mathematics, Duquesne University, Shanghai, China*)
Zheng Quan
(*Shanghai University of Science and Technology, Shanghai, China*)

Abstract

The equilibrium strategy for N-person differential games can be obtained from a min-max problem subject to differential constraints. The differential constraints can be treated by the duality and penalty methods and then an unconstrained problem can be obtained. In this paper we develop methods applying the finite element methods to compute solutions of linear-quadratic N-person games using duality and penalty formulations.

The calculations are efficient and accurate. When a (4,1)-system of Hermite cubic splines are used, our numerical results agree well with the theoretical predicted rate of convergence for the Lagrangian. Graphs and numerical data are included for illustration.

§1. Introduction

As in Part I and Part II, we consider an N-person differential game with the following dynamics:

$$(DE) \equiv \dot{x}(t) - A(t)x(t) - \sum_{i=1}^{N} B_i(t)u_i(t) - f(t) = 0, \qquad \text{on } [0,T],$$
$$x(0) = x_0 \in R^n. \tag{1.1}$$

The matrix and vector functions $A(t), f(t), B_i(t), u_i(t), i = 1, \cdots, N$, satisfy the same conditions as in Part I and II ([6] and [7]). Each player wants to minimize his cost

$$J_i(x,u) = J_i(x, u_1, \cdots, u_N), \quad i = 1, \cdots, N. \tag{1.2}$$

* Received December 31, 1988.
1) Supported in part by NSF Grant MCS 81-01892 and NASA contract No. NAS1-15810.

Let

$$F(x,u;X,v) = F(x,u_1,\cdots,u_N;x^1,\cdots,x^N,v_1,\cdots,v_N)$$
$$= \sum_{i=1}^{N}\left[J_i(x,u) - J_i(x^i,v^i)\right], \tag{1.3}$$

where $X = (x^1,\cdots,x^N)$, $v^i = (u_1,\cdots,u_{i-1},v_i,u_{i+1},\cdots,u_N)$ and each x^i is the solution of

$$(DE)_i \equiv \dot{x}^i(t) - A(t)x^i(t) - \sum_{j\neq i} B_j(t)u_j(t) - B_i(t)v_i(t) - f(t) = 0, \quad \text{on } [0,T],$$
$$x^i(0) = x_0, \qquad i = 1,\cdots,N. \tag{1.4}$$

Following [6] and [7], we consider the primal and dual problems:

(P) $\inf_{x,u}\sup_{X,v}\{F(x,u;X,v)|(x,u) \in H_n^1 \times U$ subject to (1.1), $(X,v) \in [H_n^1]^N \times U$ subject to (1.4), $i = 1,\cdots,N\}$

(D) $\sup_{p_0\in L^2}\inf_{p\in[L^2]^N} L(p_0,p)$, where $L(p_0,p) = L(p_0,p_1,\cdots,p_N) = \inf_{x,u}\sup_{X,v} L(p_0,p;x,u;X,v)$ with the Lagrangian $L : L^2x[L^2]^N \times H_n^1 \times U \times [H_n^1]^N \times U$ defined by

$$\mathbf{L}(p_0,p;x,u;X,v) \equiv F(x,u;X,v) + \left\langle p_0, \dot{x} - Ax - \sum_{j=1}^{N} B_ju_j - f\right\rangle$$
$$+\sum_{i=1}^{N}\left\langle p_i, \dot{x}^i - Ax^i - \sum_{j\neq i} B_jU_j - B_iv_i - f\right\rangle \tag{1.5}$$

for x, X satisfying $x(0) = x_0, X(0) = X_0 = (x_0,\cdots,x_0)$. We inherit the notations $U = \prod_{i=1}^N U_i$ with $U_iL_m^2(0,T)$ from Part I, and the notations of L^2 and Sobolev spaces H_n^k, H_{0n}^1 and H_{n0}^1 are the same as in [6] and [7]. We sometimes denote $L^2 = L^2(0,T)$ without mention of dimensions.

In this paper, we consider the linear quadratic problem whose cost functionals are given by

$$J_i(x,u) = \frac{1}{2}\int_0^T [|C_i(t)x(t) - z_i(t)|^2_{R^{k_i}} + \langle M_i(t)u_i(t), u_i(t)\rangle_{R^{m_i}}]dt, \tag{1.6}$$
$$i = 1,\cdots,N, \quad (x,u) \text{ feasible}$$

just as in [6], [7]; here we assume that $C_i(t)$ and $M_i(t)$ are matrix-valued functions of appropriate sizes and smoothness, and $z_i(t)$ is a vector-valued function, Furthermore, $M_i(t)$ induces a linear operator $M_i : L^2_{m_i} \to L^2_{m_i}$ which is positive definite:

$$\langle M_iu_i, u_i\rangle_{L^2_{m_i}} \geq \mu\|u_i\|^2_{L^2_{m_i}}, \quad 1 \leq i \leq N, \text{ for some } \mu > 0. \tag{1.7}$$

In §2, we formally derive the matrix Riccati equation from the duality point of view. §3 is devoted to error estimates and numerical computations. We prove sharp error bounds using the Aubin-Nitche trick. We finally present in §4 some numerical