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# DETECTING CODIMENSION TWO BIFURCATIONS WITH A PURE IMAGINARY PAIR AND A SIMPLE ZERO EIGENVALUE*[1)]

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**Abstract**

An extended system for codimension two bifurcation with a pure imaginary pair and a simple zero eigenvalue is proposed . Its regularity is proved. An efficient algorithm for solving the extended system is constructed. Finally, some results on the axial dispersion problem in a tubular non-adiabatic reactor is given.

## §1. Introduction

We consider a nonlinear evolution problem with two parameters

$$\frac{dx}{dt} = F(\lambda, \mu, x)$$

where $\lambda, \mu$ are real parameters, $x \in X$, a Hilbert space and $F$ is a smooth mapping from $R \times R \times X$ to $X$.

As well known, codimension two bifurcations with a pure imaginary pair and a simple zero eigenvalue imply that chaotic motions may happen nearby. The unfoldings of these local bifurcations contain secondary global bifurcations involving homoclinic orbits (see [1] for details.)

The following assumptions are made in the paper:

(1) There is a solution family of $x = x(\lambda, \mu)$ near $\lambda^0, \mu^0$ with $x^0 = x(\lambda^0, \mu^0)$;

(2) The Frechet derivative $F_x(\lambda, \mu, x(\lambda, \mu))$ has a pair of simple complex conjugate eigenvalues

$$r(\lambda, \mu) = u(\lambda, \mu) \pm iw(\lambda, \mu)$$

and a simple real eigenvalue $z(\lambda, \mu)$. We have

$$u(\lambda^0, \mu^0) = 0, \quad w(\lambda^0, \mu^0) = w^0 > 0, \quad z(\lambda^0, \mu^0) = 0.$$

The eigenvector corresponding to $w^0 i$ is $\phi_1^0 + i\phi_2^0$ and the real eigenvector corresponding to $z(\lambda, \mu)$ is $\phi(\lambda, \mu)$ with $\phi(\lambda^0, \mu^0) = \phi^0$. Let

$$\ker F_x^0 = \{c\phi^0 |\ c \in \mathbb{R}\}.$$

There exists a nontrivial $\psi^0 \in X$ such that Range $F_x^0 = \{x \in X | \langle \psi^0, x\rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ is an inner product.

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(3) $F^0_\lambda \notin \text{Range}\, F^0_x$, i.e. $\langle \psi^0, F^0_\lambda \rangle \neq 0$;
(4) $F^0_x$ has no eigenvalues of the form $kiw^0, k \neq \pm 1$.

There are two ways for detecting the codimension two bifurcations. One way is to find the cross point of the Hopf bifurcation branch and the folds branch with respect to when $\mu$ varies. We can get these branches by using the algorithms in [2] and [3]. The other way is to determine the codimension two bifurcations directly. In Section 2, we propose an extended system for the codimension two bifurcations, which is regular. An efficient algorithm for solving the extended system is given in Section 3. Finally, we use the above methods to get the codimension two bifurcations in the axial dispersion problem in a tubular non-adiabatic reactor.

## §2. An Extended System

We propose an extended system for the codimension two bifurcations as follows:

$$G(y) = \begin{bmatrix} F(\lambda,\mu,x) \\ F_x(\lambda,\mu,x)\phi \\ e\phi - 1 \\ [F_x^2(\lambda,\mu,x) + w^2 I]p \\ \langle p,p\rangle - 1 \\ \langle q,p\rangle \end{bmatrix} = 0 \tag{2.1}$$

where $y = (\lambda,\mu,w,x,\phi,p)$, $q$ is a constant vector with nonzero projection on span $\{\phi_1^0,\phi_2^0\}$, $e \in X^*$ is chosen later on.

There is a unique vector $p^0 \in$ Span $\{\phi_1^0,\phi_2^0\}$ such that $y^0 = (\lambda^0,\mu^0,w^0,x^0,\phi^0,p^0)$ is the isolated solution of (2.1). The Frechet derivative $G_y(y^0)$ is

$$\begin{bmatrix} F_x^0 & 0 & 0 & F_\lambda^0 & F_\mu^0 & 0 \\ F_{xx}^0\phi^0 & F_x^0 & 0 & F_{\lambda x}^0\phi^0 & F_{\mu x}^0\phi^0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ F_{xx}^0F_x^0p^0 + F_x^0F_{xx}^0p^0 & 0 & F_x^{0^2} + w^{0^2}I & F_{\lambda x}^0F_x^0p^0 + F_x^0F_{\lambda x}^0p^0 & F_{\mu x}^0F_x^0p^0 + F_x^0F_{\mu x}^0p^0 & 2w^0p^0 \\ 0 & 0 & 2p^0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \end{bmatrix}$$

Let

$$N([(F_x^0)^2 + w^{0^2}I]^*) = \text{Span}\,\{\psi_1^0,\psi_2^0\}$$

with

$$\langle \psi_i^0, \phi_j^0 \rangle = \delta_{ij}, \qquad i,j = 1,2 .$$

We can get

$$\psi_3^0 = (d_1^2 + d_2^2)^{-1}(d_1\psi_1^0 + d_2\psi_2^0), \qquad \psi_4^0 = (d_1^2 + d_2^2)^{-1}(d_2\psi_1^0 - d_1\psi_2^0)$$

with $\langle \psi_3^0, p^0 \rangle = 1$, $\quad \langle \psi_4^0, p^0 \rangle = 0$, where $p^0 = d_1\phi_1^0 + d_2\phi_2^0$ . Let $g_1; g_2$ be respectively the unique solution of

$$\begin{cases} F_x^0 g_1 = QF_\lambda^0, \\ eg_1 = 0; \end{cases} \qquad \begin{cases} F_x^0 g_2 = QF_\mu^0, \\ eg_2 = 0 \end{cases}$$