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INVERSE SPECTRUM PROBLEMS FOR BLOCK JACOBI MATRIX*

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Abstract

By establishing the spectrum (matrix) function for the block Jacobi matrix, theorems of existence and uniqueness for the inverse problem and algorithms for its solution are obtained. The study takes into account all possible multiple-eigenvalue cases that are very difficult to deal with by other means.

1. Introduction

There is an extensive study [1–5] on inverse eigenvalue problems for Jacobi matrices, but only a few papers [4] deal with block or banded matrices which arise more often in practice. In these papers most work is restricted to the case of simple eigenvalue only, which is often not the case in practice. Further study of such problems involving multiple eigenvalues is urgently needed.

In this paper, we study the Jacobi matrix with entries as $r \times r$ matrices of the form:

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ b_1^* & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & b_{n-1} \\ & & & b_{n-1}^* & a_n \end{bmatrix},$$

$a_j,\ b_j\ \ r \times r$ matrices on C, $a_j = a_j^*$, all b_j invertible.

We denote the set of all $r \times r$ matrices on C as a ring F. The main points we must bear in mind are

(1) for $a, b \in F$, then $ab \neq ba$ in general,

(2) for $a \in F$ and $a \neq 0, a^{-1}$ may not exist in general.

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First, we need to establish operations in F, and some definitions. Then is Section 2 we will apply the Fourier theory to the matrix A. This fresh approach will turn out to be very useful. In Section 3. a general inverse spectrum problem and the main theorems of the problem will be given. Two algorithms based on G-L theory are formulated, and another algorithm based on orthogonalization is given in Section 4. Finally, a brief description of the applications and numerical testing will be presented. The main advantage of our treatment is that; our conclusion takes into account all possible multiple-eigenvalue cases that are difficult to deal with by other means.

Beside the well known operations such as multiplication, scalar multiplication and addition, we recall the conjugate operation '$*$' in F:

(1) for $a \in F, a^* = \bar{a}^T, (a^*)^* = a,$ (2)

(2) for $a, b \in F, (ab)^* = b^*a^*,$

(3) for $a \in F, a^*a$ and aa^* are semi-positive and self-adjoint,

(4) $a = 0 \leftrightarrow a^*a = aa^* = 0.$

Let H denote a linear space of n-dimensional column vectors on F, that is

$$H = \{f = (f_1, f_2, \cdots, f_n)^T, \text{ all } f_K \in F\}.$$

And denote H^* as its conjugate space, that is

$$H^* = \{f^* = (f_1^*, f_2^*, \cdots, f_n^*), \text{ all } f_K \in F\}.$$

Then we define the inner product of $f, g \in H$ as follows:

(1) $(f, g) = f^*g = f_1^*g_1 + f_2^*g_2 + \cdots + f_n^*g_n \in F,$ (3)

(2) $(f, g) = (g, f)^*,$

(3) $(f, f) = f^*f \geq 0,$

(4) $f = 0 \leftrightarrow (f, f) = 0.$

Denote by 0 and I the zero element and identity element respectively of either F or H or some matrices without causing confusion.

Since (f, g) is a matrix in F, then it is natural to regard $a^*a(aa^*)$ in (2) and (f, f) in (3) as certain matrix measurement instead of a usual norm. This is something interesting beyond our problem.

2. Fourier Theory for a Block Jacobi Matrix

First, we introduce an eigenfunction $\phi(\lambda), \lambda \in (-\infty, +\infty)$, for (1) as follows:

$$\phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda), \cdots \phi_n(\lambda))^T, \phi(\lambda) = I,$$

$$A\phi(\lambda) = \lambda\phi(\lambda) + R(\lambda), \quad R(\lambda) = (0, 0, \cdots, \Phi_n(\lambda))^T \tag{4}$$

where $\Phi_n(\lambda) = b_{n-1}^*\phi_{n-1}(\lambda) + (a_n - \lambda)\phi_n(\lambda)$. In general $\Phi_n(\lambda) \neq 0$, unless when λ is an eigenvalue, such that det $|\Phi_n(\lambda)| = 0$ and the homogeneous system $\Phi_n(\lambda)\gamma = 0$ has nontrival solutions $\gamma = \gamma(\lambda)$.