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STREAMLINE DIFFUSION METHODS FOR OPERATOR EQUATIONS*

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Abstract

To solve a class of operator equations numerically, some general streamline diffusion methods with satisfactory convergence properties are presented in this paper. It is proved that the approximation accuracy is only half a power of h, the mesh size, from being optimal when these methods are applied to mixed problems and convection-diffusion problems.

§1. Introduction

It was observed long ago that usual finite element methods for certain equations such as mixed and hyperbolic equations do not work well or even do not give reasonable results. It has to be of particular concern since in many interesting practical problems, well-approximations are desired. Trying to solve the problems, we present in this paper some streamline diffusion methods. It is shown that these methods work well and possess satisfactory convergence, for a class of operator equations, including mixed and hyperbolic types.

These methods are based upon an idea first introduced by Raithby[9] for finite difference methods and by Hughes and Brooks[7] for finite element methods for special problems in fluid dynamics by adding an artificial diffusion term acting only in the direction of the streamlines. This idea was also taken up by Johnson[8] with observation that such a streamline diffusion term can be introduced very naturally in the standard Galerkin method without modifying the equation. We refer to a series of work by Johnson[8] and his cooperators (see the literature cited therein) and by the author[10] in this direction.

This paper consists of four sections. In Section 2, general streamline diffusion methods are introduced, and existence, uniqueness and error analysis for these methods are discussed. In Section 3, some error estimates with half a power of h from being optimal are then given in the case of a boundary value problem of mixed type. In the last section, analogous results for convection-diffusion problems are presented.

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We shall point out that these methods are somewhat different from those stated in [8] and [10] when applied to convection-diffusion equations.

§2. Operator Equations

Suppose that W and X are two Hilbert spaces. We consider the following operator equation:

$$Lu = f, \tag{2.1}$$

where L is a linear operator from W into X, and $f \in X$.

Let $\{W^h : h \in (0,1)\}$ be a family of finite-dimensional subspaces of W. An approximation solution $u_h \in W^h$ of the problem (2.1) will be defined by the following discrete scheme:

Given $f \in X$, find $u_h \in W^h$ such that

$$A(u_h, v) = (f, lv + hLv), \forall v \in W^h, \tag{2.2}$$

where $A(u,v) = (Lu, lv + hLv)$, $l : W \to X$ is a linear operator and $(\cdot,\cdot)$ is the inner product of X.

Now we take the following assumption:

Assumption I. There exists a Hilbert space $(Y, \|\cdot\|_Y)$, into which W is imbedded boundedly, such that

$$(Lu, lu) \geq c\|u\|_Y^2, \ \forall u \in W. \tag{2.3}$$

If we set

$$|||u|||^2 = \|u\|_Y^2 + h\|Lu\|_X^2,$$

then (2.3) implies

$$A(u,u) \geq c|||u|||^2, \ \forall u \in W. \tag{2.4}$$

Let $\{\phi_i\}_{i=1}^n$ be a basis for W^h and

$$u_h = \sum_{i=1}^n z_i \phi_i.$$

Denote $z \doteq (z_1, \cdots, z_n)^T$ and $b = (b_1, \cdots, b_n)^T$ with

$$b_i = (f, l\phi_i + hL\phi_i).$$

Then u_h is given by the linear system:

$$\tilde{A}z = b,$$

where $\tilde{A} = (a_{ij})_{1 \leq i,j \leq n}$ and $a_{ij} = A(\phi_i, \phi_j)$, $1 \leq i,j \leq n$.

Lemma 2.1. *$\tilde{A}$ is invertible.*

Proof. Suppose that $\tilde{A}z = 0$ for some vector z. Setting $z_h = \sum_{i=1}^n z_i \phi_i$, we find that

$$o = z^T \tilde{A} z = A(z_h, z_h) \geq c\|z_h\|_Y^2, \tag{2.5}$$