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THE HERMITE SCHEME FOR SEMILINEAR SINGULAR PERTURBATION PROBLEMS*[1)]

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Abstract

A numerical method for singularly perturbed semilinear boundary value problems is given. The method uses the fourth order Hermite scheme on a special discretization mesh. Its stability and convergence are investigated in the discrete L^1 norm.

§1. Introduction

We shall consider the following singularly perturbed boundary value problem:

$$T_u := -\varepsilon^2 u'' + c(x,u) = 0,\ x \in I = [0,1],$$

$$u(0) = u(1) = 0, \tag{1}$$

where $\varepsilon \in (0, \varepsilon^*]$ (usually $\varepsilon^* << 1$). Throughout the paper we shall assume:

$$c \in C^6(I \times I\!R), \tag{2.a}$$

$$c_u(x,u) > \gamma^2,\ (x,u) \in I \times I\!R,\ \gamma > 0. \tag{2.b}$$

These conditions guarantee that the problem (1) has a unique solution u_ε, $u_\varepsilon \in C^B(I \times I\!R)$, which exhibits two boundary layers at the endpoints of I. In particular, the following estimates hold, see [22]:

$$|u_\varepsilon^{(k)}(x)| \le M[1 + \varepsilon^{-k}(\exp(-\gamma x/\varepsilon) + \exp(\gamma(x-1)/\varepsilon))],\ x \in I,\ k = 0(1)6, \tag{3}$$

where M does not depend on ε.

Because of such a behaviour of u_ε it is necessary to use special methods to solve the problem numerically. We shall use a finite-difference scheme on a special non-equidistant discretization mesh which is dense in the layers. The mesh will guarantee that the local truncation errors of the scheme will be uniform (by "uniform" we shall always mean "uniform in ε"); hence the discretization will be uniformly consistent

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with the continuous problem. Then the uniform convergence (the convergence of the numerical solution towards the restriction of u_ε on the mesh) will follow if we show that our discretization is uniformly stable. Usually, as in Doolan, Miller, Schilders [7], Herceg, Vulanović [13], Herceg [8, 9, 10], Herceg, Petrović [12], Herceg, Vualnović, Petrović [14], Vulanović, Herceg, Petrović [26], Vulanović [21, 22, 23], the stability is shown in the maximum norm; hence the pointwise uniform convergence follows. The order of the convergence depends on the scheme used. Higher order convergence was proved in Vulanović, Herceg, Petrović [26], Herceg, Vulanović, Petrović [14] and Herceg [9], while Herceg [10], and Herceg, Petrović [12] used higher order schemes in the layers only. These papers, as well as Vulanović [22], Herceg, Vulanović [13] and Vulanović [23], use the approach of special discretization meshes. The concept of exponential fitting was used in Doolan, Miller, Schilders [7], Herceg [8], Vulanović [21], and Vulanović [23]. The method from [3] is based on piecewise linear interpolation, and for the use of spline-difference schemes see Surla [16, 17, 18, 19]. For other papers which deal with the numerical solution of the problem (1), see Herceg [9].

In this paper we shall use a discretization of the same type as in Herceg [9], Herceg, Vulanović, Petrović [14]. Basically, the Hermite scheme is used, but at some mesh points it is replaced by the standard central scheme. Such a switch is used in order to prove the uniform stability. For the same reason Herceg [9] and Herceg, Vulanović, Petrović [14] have a restriction on the nonlinearity of $c(x,u)$. Essentially, the following is required:

$$c_u(x,u) \le \Gamma, \; x \in I, \; u \in I\!R; \quad 5\gamma^2 - 2\Gamma > 0. \tag{4}$$

Obviously, such an assumption is unpleasant, and our aim here will be to avoid it. We shall prove the uniform stability in the discrete L^1 norm (cf. Vulanović [24], [25] where this norm was used for discretizations of quasilinear singular perturbation problems) and for this (4) is not needed. Such a result was announced in Herceg [9] and Herceg, Vulanović, Petrović [14].

Thus we shall obtain the uniform convergence in the discrete L^1 norm. The L^1-error will be estimated by

$$M[\varepsilon n^{-4} + n^{-1}\exp(-pn)]$$

where n is the number of mesh steps, p is a positive constant independent of n and ε, and throughout the paper M denotes a positive generic constant independent of n and ε. From this we shall get that

$$M[n^{-3} + \varepsilon^{-1}\exp(-pn)]$$

is the upper bound for the maximal pointwise error. This is worse than Mn^{-4} from Herceg [9]. However, we point out that the numerical method which will be given here is essentially the same as the method from Herceg [9] (the deferent pointwise error estimates result from the different norms used); hence we might expect the uniform fourth order pointwise convergence to be still present. Our numercical experiments confirm that.