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A FAMILY OF VISCOSITY SPLITTING SCHEME FOR THE NAVIER-STOKES EQUATIONS*

Zhang Ping-wen
(*Department of Mathematics, Peking University, Beijing, China*)

Abstract

In the paper, a family of viscosity splitting method is introduced for solving the initial boundary value problems of Navier-Stokes equation. Some stability and convergence estimates of the method are proved.

§1. Introduction

Since the publication of Chorin's work in 1973, the convergence problem of viscous splitting for the Navier-Stokes equation has been considered by several authors. Beale and Majda proved a convergence theorem for the Cauchy problems. Chorin, Hughes, McCracken and Marsden suggested a product formula for the initial boundary value problem, without convergence proof, follows:

$$u_n(t) = (H(\frac{t}{n}) \circ \phi \circ E(\frac{t}{n}))^n u_0 \tag{1.1}$$

where $H(\cdot)$ is the Stokes solver, $E(\cdot)$ is the Euler solver and ϕ is a so called "vorticity creation operator", the capacity of which is to maintain the no-slip condition at the surface. Ying Long-an considered this scheme and proved that (1.1) does not converge; he also proved that if a nonhomogeneous term is added to the Stokes equation to neutralize the error arising from the operator ϕ, then this scheme converges, the rate of convergence is $O(k)$ in $L^\infty(0,T;(H^1(\Omega))^2)$ for the two dimensional case, and $O(k)$ in $L^\infty(0,T;(L^2(\Omega))^3)$ for the three dimensional case, where k is the length of time step. Alessandrini, Douglis and Fabes also considered the initial boundary value problems and proved the convergence of the scheme

$$u_n(t) = (H(\frac{t}{n}) \circ E_M(\frac{t}{n}))^n u_0 \tag{1.2}$$

where $E_M(\cdot)$ is an approximate Euler solver with the solutions of the Euler equation replaced by polynomials. Zheng and Huang considered a scheme similar to (1.2), where there is also no operator ϕ, but $E_M(\cdot)$ is replaced by $E(\cdot)$; they proved that the rate

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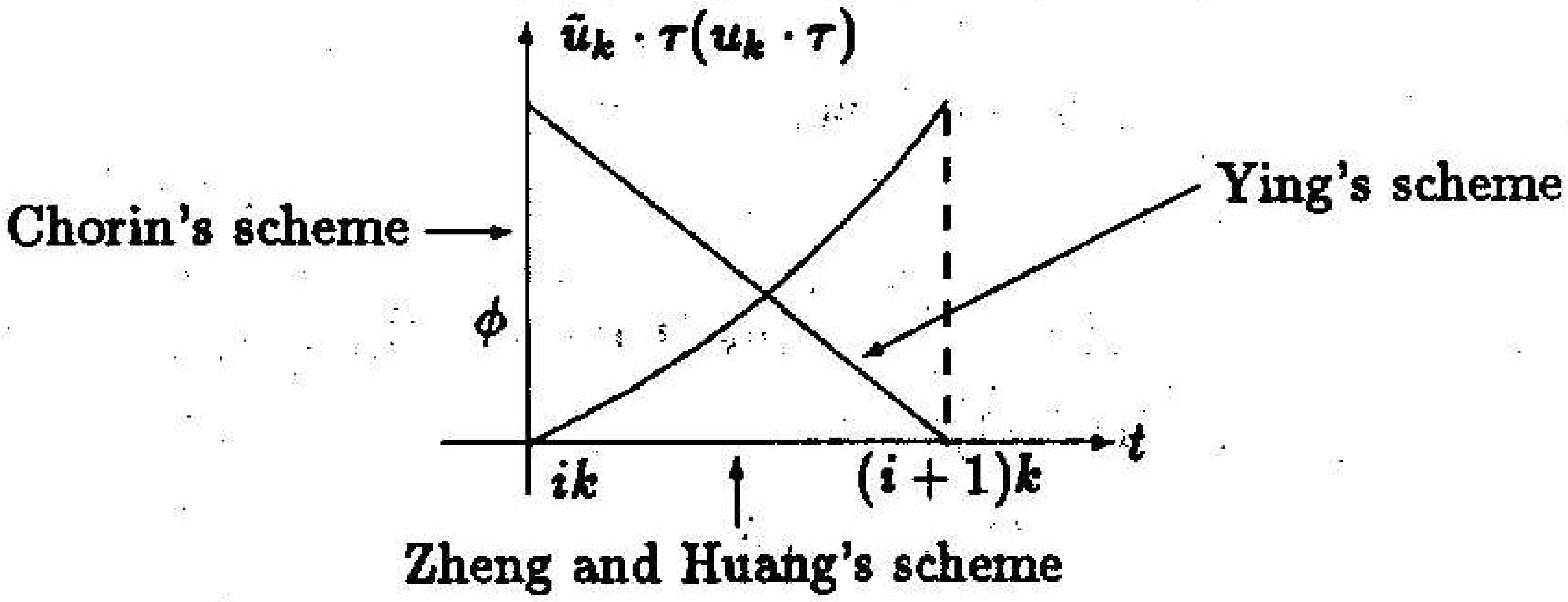
of convergence for the two dimensional case is $O(k^{\frac{3}{4}-\varepsilon})$ in $L^\infty(0,T;(L^2(\Omega))^2)$, where $0<\varepsilon<1/4$. Recently, Ying Long-an considered a scheme

$$u_n(t)=(\widehat{H}(\frac{t}{n})\circ E(\frac{t}{n}))^n u_0 \tag{1.3}$$

where $\widehat{H}(\cdot)$ is the Stokes solver with nonhomogeneous on boundary conditions; he proved that the rate of convergence for the two dimensional case is $O(k)$ in $L^\infty(0,T;(H^1(\Omega))^2)$.

To understand those schemes clearly, let us give a chart.



In the chart, $\tilde{u}_k$ are the solutions of the Euler equations, u_k are the solutions of the Stokes equations, and τ is the tagent vector.

The purpose of this paper is to study a family of viscosity splitting scheme similar to (1.3). We will prove a convergence theorem where the rate of convergence for the two-dimensional case is $O(k^{\frac{1}{4}-\varepsilon})$ in $L^\infty(0,T;(H^1(\Omega))^2)$, where $0<\varepsilon<1/4$. For simplicity, we only consider simply connected bounded domains in R^2.

§2. The Scheme and the Main Theorem

Let $x=(x_1,x_2)$ and $y=(y_1,y_2)$ be points in R^2 and Ω be a simply connected domain in R^2 with sufficiently smooth boundary $\partial\Omega$. The initial boundary value problem of the Navier-Stokes equation is given as

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\frac{1}{\rho}\nabla P=\nu\Delta u+f,\quad x\in\Omega,t>0, \tag{2.1}$$

$$\nabla\cdot u=0,\qquad x\in\Omega,t>0, \tag{2.2}$$

$$u|_{x\in\partial\Omega}=0, \tag{2.3}$$

$$u|_{t=0}=u_0(x) \tag{2.4}$$

where $u=(u_1,u_2)$ is the velocity, P is the pressure, and ν,ρ are positive constants. Throughout this paper we assume that the solution (u,P) of the above problem is sufficiently smooth on $\overline{\Omega}\times[0,T]$, and the usual notations $H^s(\Omega)$ and $W^{m,p}(\Omega)$ for Sobolev spaces and $\|\cdot\|_s$ and $\|\cdot\|_{m,p}$ for norms in Sobolev spaces are applied.