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ON NONLINEAR GALERKIN APPROXIMATION*

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Abstract

Nonlinear Galerkin methods are numerical schemes adapted well to the long time integration of evolution partial differential equations. The aim of this paper is to discuss such schemes for reaction diffusion equations. The convergence results are proved.

1. Introduction

In order to solve the problem of long time integration of evolution partial differential equations, nonlinear Galerkin methods are introduced in recent years. Such methods stem from the theory of inertial manifolds and approximate inertial manifolds. We recall an inertial manifold is a finite dimensional smooth manifold which contains the global attractor and attracts every orbit at an exponential rate[1,2]. However, there are still many dissipative partial differential equations for which the existence of inertial manifolds is not known; there are even in some cases nonexistence results. These problems have lead to introduce the weak concept of approximate inertial manifolds. These manifolds are finite dimensional smooth manifolds such that all orbits enter their a thin neighborhood after a certain time. The existence of such manifolds can be found in Foias, Manley and Temam [3] ; Marion [4].

The algorithms which produce an approximate solution lying on an approximate inertial manifold are introduced by Marion and Temam[5]. They have been called nonlinear Galerkin methods, opposite to the usual Galerkin method which produces an approximate solution in the linear space spanned by the first m's functions of the Galerkin basis.

The improvenments of the nonlinear Galerkin methods over the usual Galerkin method are evidenced by theoretical results and numerical computations that show a significant gain in computing time[5,6,7]. In this paper, we study such nonlinear Galerkin methods for reaction diffusion equations, and prove the convergence rusults.

In section 1, we recall some known results for reaction diffusion equations, and introduce an approximate inertial manifold Σ, where Σ is first given by Wang[8]. Section 2 contains the nonlinear Galerkin methods based on Σ and our main results. The convergence results are obtained in this section.

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2. An Approximate Inertial Manifold

We consider the following problem with a real valued function $u(x,t)$ defined on $R^+ \times \Omega$, where Ω denotes a regular bounded set of $R^n (n \leq 4)$:

$$\frac{\partial u}{\partial t} - d\Delta u + g(u) = 0 \ , \qquad \text{in } R^+ \times \Omega \tag{2.1}$$

The equation is supplemented with the initial condition

$$u(x,0) = u_0(x) \qquad \text{in } \Omega \tag{2.2}$$

and one of the three following boundary conditions :

$$\begin{cases} \text{Dirichlet} & u = 0 \text{ on } \Gamma = \partial\Omega, \\ \text{Neumann} & \frac{\partial u}{\partial \gamma} = 0 \ \text{ on } \Gamma, \\ \text{Periodic} & \Omega = \prod_{i=1}^{n} (0, L_i) \text{ and } u \text{ is } \Omega \text{ periodic.} \end{cases} \tag{2.3}$$

where $d > 0$ is a diffusion coefficient. We assume that $g \in C^1(R, R)$ satisfies

$$g'(s) \geq -c_1, \qquad \forall\, s \in R \tag{2.4}$$

$$c_2|s|^k - c_4 \leq g(s) \cdot s \leq c_3|s|^k + c_4, \quad \forall\, s \in R, \tag{2.5}$$

where $k > 2$ is a integer and $c_i > 0$ is constant.
Let $Au = -d\Delta u + u$, then A is a unbounded self-adjoint positive operator on $H = L^2(\Omega)$ with domain

$$D(A) = \{u \in H^2(\Omega) : \text{u satisfies (2.3)}\}$$

Let $|\cdot|$ be the norm of H with scalar product $(\cdot,\cdot)$; and $\|\cdot\| = |A^{\frac{1}{2}} \cdot|$ be the norm of $V = D(A^{\frac{1}{2}})$ with scalar product $((\cdot))$. Denote by $|\cdot|_p$ the norm of $L^p(\Omega)$ for $1 \leq p < \infty$.$(|\cdot|_2 = |\cdot|)$.

Since A^{-1} is compact, there exists an orthonormal basis of H consisting of eigenvectors w_j of A

$$Aw_j = \lambda_j w_j, \ j = 1, 2, \cdots$$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty \text{ as } j \to +\infty.$$

Under assumptions (2.4) and (2.5), it follows from [4] that for u_0 given in H, the problem (2.1)-(2.3) possesses a unique solution u defined on R^+ such that

$$u \in C(R^+; H) \bigcap L^2(0, T; V), \forall T > 0.$$

Furthermore, if $u_0 \in V \cap L^k(\Omega)$, then

$$u \in C(R^+; H) \bigcap L^2(0, T; D(A)), \ \forall T > 0.$$