Journal of Computational Mathematics, Vol.16, No.6, 1998, 561–570.

ON THE COUPLING OF BOUNDARY INTEGRAL AND FINITE ELEMENT METHODS FOR SIGNORINI PROBLEMS*

Wei-jun Tang Hong-yuan Fu Long-jun Shen
(*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, China*)

Abstract

In this paper, a exterior Signorini problem is reduced to a variational inequality on a bounded inner region with the help of a coupling of boundary integral and finite element methods. We established a equivalence between the original exterior Signorini problem and the variational inequality on the bounded inner region coupled with two integral equations on an auxiliary boundary. We also introduce a finite element approximation of the variational inequality and a boundary element approximation of the integral equations. Furthermore, the optimal error estimates are given.

Key words: Boundary element, finite element, Signorini problems.

1. Introduction

Partial differential equations subject to unilateral boundary conditions are usually called Signorini problems in the literature. These problems have been studied by many authods since the appearence of the historical paper by A. Signorini in 1933 [25]. Signorini problems arose in many areas of applications e.g., the elasticity with unilateral conditions[10], the fluid mechnics problems in media with semipermeable boundaries[8,12], the electropaint process[1] etc. For the existence, uniqueness and regularity results for Signorini type problems we refer the reader to [3, 11]. Furthermore, the numerical solution of the Signorini problems by the finite element method has been discussed[4,13]. Boundary element method for solving Signorini problems has been presented in [14, 15].

In this paper, we will present a coupling of boundary integral and finite element methods for solving a exterior Signorini problem, which is reduced to a equivalent new variational inequality on a bounded inner region coupled with two integral equations on an auxiliary boundary. The bilinear form arising in this variational inequality is continuous and coercive on suitable subspaces of some Sobolev space. This leads to existence and uniqueness for the solution of the variational inequality. Furthermore, a coupling of boundary element and finite element methods for numerical solution of the variational inequality is proposed and optimal error estimates are derived.

* Received June 6, 1996.

2. The New Variational Inequality on a Bounded Inner Region

Let Ω^c be the complement of a bounded regular region in R^2 with boundary Γ. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$ (as shown in Fig.1), with $\Gamma_0 \cap \Gamma_1 = \phi$, $\Gamma_0 \neq \phi$, we consider the following Signorini problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega^c, \\ u = 0, & \text{on } \Gamma_0, \\ u \geq 0, \ \dfrac{\partial u}{\partial n} \geq 0, & \text{on } \Gamma_1, \\ u\dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_1, \\ u \text{ is bounded}, & \text{when } \ |x| \to \infty, \end{cases} \tag{2.1}$$

Fig. 1

where f has its support in a bounded subregion Ω_1 of Ω^c. In case of $\Gamma_0 = \phi$, f satisfies a compatibility condition[8]

$$\int_{\Omega_1} f dx \geq 0. \tag{2.2}$$

Let $\Omega_2 = \Omega^c \backslash \bar{\Omega}_1$, $\Gamma_2 = \partial\Omega_2$ (see Fig.2). We will solve the exterior Signorini problem (2.1) by using the coupling of boundary element and finite element methods. Consider the equivalent system of Signorini problem:

$$\begin{cases} -\Delta u_1 = f, & \text{in } \Omega_1, \\ -\Delta u_2 = 0, & \text{in } \Omega_2, \\ u_1 = u_2, & \text{on } \Gamma_2, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \sigma, & \text{on } \Gamma_2, \\ u_1 = 0, & \text{on } \Gamma_0, \\ u_1 \geq 0, \ \dfrac{\partial u_1}{\partial n} \geq 0, & \text{on } \Gamma_1, \\ u_1\dfrac{\partial u_1}{\partial n} = 0, & \text{on } \Gamma_1, \\ u_2 \text{ is bounded}, & \text{when } \ |x| \to \infty, \end{cases} \tag{2.3}$$

Fig. 2

where $u_i = u|_{\Omega_i}$, $i = 1, 2$, and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative to the boundary $\partial\Omega_1 = \Gamma \cup \Gamma_2$ (see Fig.2).

We note that u_1 will be completely determined, if σ is known. On the other hand, the function σ depends linearly on $u_1|_{\Gamma_2}$ via the solution of the following exterior boundary value problem

$$\begin{cases} -\Delta u_2 = 0, & \text{in } \Omega_2, \\ u_2 = u_1, & \text{on } \Gamma_2, \\ u_2 \text{ is bounded}, & \text{when } |x| \to \infty. \end{cases} \tag{2.4}$$

Hence the next step in the coupling procedures is to derive equations on Γ_2 which