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THE DEFECT ITERATION OF THE FINITE ELEMENT FOR ELLIPTIC BOUNDARY VALUE PROBLEMS AND PETROV-GALERKIN APPROXIMATION*1)

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Abstract

In this paper we introduce a Petrov-Galerkin approximation model to the solution of linear and semi-linear elliptic boundary value problems in which piecewise quadratic polynomial space and piecewise linear polynomial space are used as the shape function space and the test function space, respectively. We prove that the approximation order of the standard quadratic finite element can be attained in this Petrov-Galerkin model. Based on the so-called "contractivity" of the interpolation operator, we further prove that the defect iterative sequence of the linear finite element solution converge to the proposed Petrov-Galerkin approximate solution.

Key words: Petrov-Galerkin approximation, defect iteration correction, interpolation operator

1. Introduction

Frank etc. cf. [1] established the iterated defect correction scheme for finite element of elliptic boundary problems. For linear elliptic boundary value problem [2–5] have discussed the efficiency of the scheme by using superconvergence and asymptotic expansion under the conditions that the partition is uniform or strongly regular. It is proven that for the given linear finite element solution as initial approximation the first iterated correction can achieve the approximation order that the standard quadratic finite element solution has. However, for example, when the partition is only piecewise uniform, the approximation order of the quadratic finite element can not be obtained by the first iterated correction under the natural smoothness assumption. Moreover

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numerical results present that the approximation order is lower around the crossnode of bigger element; cf. [2, 3], although the exact solution is sufficiently smooth. On the other hand, the results in [2, 3] point out that the iterated corrections after many times can make up for a loss of approximation defect. That is, the iterated defect correction scheme is efficient. How can one give a theorectical explanation?

For the linear two-point boundary problem, it has been shown [7] that the iterated defect correction of finite element solution converges to the Petrov-Galerkin approximation solution. Can one further study the convergence of the iterated defect correction scheme for the finite element of the elliptic boundary problem by the aid of ideas given in [7]? To answer the question, we should establish a suitable Petrov-Galerkin scheme for the elliptic boundary problem. Although the theoretical analysis for Petrov-Galerkin approxiamtion of the elliptic boundary problem has been established in [8], to construct a practicable Petrov-Galerkin scheme and prove its convergence and error estimation is important work. This paper will be dedicated to this problem.

The remainder of the paper is organized as follows. We establish the so-called contractivity (cf. Theorem 2.1) of the interpolation operator in the next section. Then, in Section 3 we consider the linear elliptic boundary problem and establish a scheme of Petrov-Galerkin approximation. Furthermore we prove that the iterated defect correction for the linear finite element solution geometrically converges to the solution of the proposed Petrov-Galerkin scheme. Finally, in Section 4 we report the similar results as in Section 3 for the semi-linear elliptic boundary problem.

2. Approximation property of interpolation operator

Given a triangle T with vertices at $P_i = (x_i, y_i)$, $i = 1, 2, 3$, denote by Δ the area of T and set

$$\begin{aligned} \xi_1 = x_2 - x_3,\ \xi_2 = x_3 - x_1,\ \xi_3 = x_1 - x_2 \\ \eta_1 = y_2 - y_3,\ \eta_2 = y_3 - y_1,\ \eta_3 = y_1 - y_2 \end{aligned} \tag{2.1}$$

$$r_1(T) = \frac{1}{\Delta}(\xi_2\xi_3 + \eta_2\eta_3), \quad r_2(T) = \frac{1}{\Delta}(\xi_3\xi_1 + \eta_3\eta_1),\ r_3(T) = \frac{1}{\Delta}(\xi_2\xi_3 + \eta_2\eta_3),$$

$$t_1(T) = \frac{1}{\Delta}(\xi_1^2 + \eta_1^2),\ t_2(T) = \frac{1}{\Delta}(\xi_2^2 + \eta_2^2),\ t_3(T) = \frac{1}{\Delta}(\xi_3^2 + \eta_3^2) \tag{2.2}$$

$$l_1(T)^2 = \xi_1^2 + \eta_1^2,\ l_2(T)^2 = \xi_2^2 + \eta_2^2,\ l_3(T)^2 = \xi_3^2 + \eta_3^2, \tag{2.3}$$

where l_i is the length of the edge $P_{i-1}P_{i+1}$ opposite to the vertex P_i (with $i = 1, 2, 3$ and $i-1, i, i+1 \in Z_3$ similarly defined in the following) and it is obvious that $r_i(T) \leq 0$ and

$$t_1(T) = -r_2(T) - r_3(T),\ t_2(T) = -r_3(T) - r_1(T),\ t_3(T) = -r_1(T) - r_2(T) \tag{2.4}$$