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SPECTRAL AND PSEUDOSPECTRAL APPROXIMATIONS IN TIME FOR PARABOLIC EQUATIONS*[1)]

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Abstract

In this paper, spectral and pseudospectral methods are applied to both time and space variables for parabolic equations. Spectral and pseudospectral schemes are given, and error estimates are obtained for approximate solutions.

Key words: Spectral approximation, pseudospectral approximation, parabolic equation, error estimate

1. Introduction

In recent years, it has been shown that spectral methods are very useful to solve partical differential equations. Spectral methods, in which the approximate solution is a polynomial of high degree, are known to be very accurate when the solution to be approximated is very smooth (see [2] for details). Using spectral methods to time-dependent partical differential equations, a standard scheme is done in space only, while finite difference is done in time (the same to finite element method, too). Hence, no matter how smooth the exact solution is, in general, the error order in time can not be raised. The error in time decide the global error of the approximate solution. Many efforts have been made on the discretization in time, for instance, in [6] and [7] discontinuous Galerkin method in time is studied for parabolic equations. Recently, I. Babuska and T. Janik[3] discussed the p-version of finite element method in time for parabolic equations. In [4] and [5] H.T. Ezer has proposed spectral methods in time using polynomial approximation of the evolution operator in Chebyshev least-squares sense for parabolic equations and hyperbolic equations. In this paper, for convenience we use the spectral methods in both space and time variables. If we use the finite element method in space, some parallel conclusions can also been obtained.

2. Variational Principle

Let $I = (-1, 1)$, $D = [0, 2\pi]$, $Q = D \times I$. For convenience we consider the following model problem

$$u_t - u_{xx} + u = f(x,t), \quad \text{in } Q \tag{2.1}$$
$$u(x,t) = u(x + 2\pi, t), \tag{2.2}$$
$$u(x,-1) = g(x). \quad \text{in } D \tag{2.3}$$

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Remark 1. If $f \in H^{k,\frac{k}{2}}(Q)$, $g \in H_p^{k+2}(D)$, from regularity of solutions of parabolic equations, the solution $u(x,t)$ of (2.1)–(2.3) is in $H^{k+2,\frac{k}{2}+1}(Q)$.

Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote respectively the inner product and the norm in $L^2(I)$, $H_p^m(D)$ denote the m order periodic Sobolev space with the norm $\|u\|_m$, $X = L^2(I; H_p^1(D))$ with the norm $\|u\|_X = \Big(\int_I \|u\|_1^2 dt\Big)^{\frac{1}{2}}$, $C_p^0 = \{v \in C^\infty(Q) \,|\, v(x,t) \in C_p^\infty(D),\ \forall t \in I,\ v(x,1) = 0\}$, and Y denote the complete space of C_p^0 with respect to the norm $\|v\|_Y = \Big(\int_I (\|v_t\|^2 + \|v\|_1^2) dt\Big)^{\frac{1}{2}}$, where

$$\|v_t\| = \sup_{z \in H_p^1(D)} \frac{|\int_D v_t z dx|}{\|z\|_1}.$$

Let us define on $X \times Y$ the bilinear form

$$B(u,v) = \int_I \int_D (-u\overline{v}_t + u_x \overline{v}_x + u\overline{v}) dx dt, \quad \forall u \in X, \quad v \in Y.$$

Let $F \in Y'$. We consider the following variational problem P: find u_0 in X such that

$$B(u_0, v) = F(v), \quad \forall v \in Y. \tag{2.4}$$

It is similar to problem P in [3] in proof, we obtain theorem 1 for the problem P.

Theorem 1. *Problem P has a unique solution u_0 in X and there exists a constant C independent of u_0 and F such that*

$$\|u_0\|_X \le C\|F\|_{Y'}.$$

Proof. Let $\lambda_j^2 = j^2 + 1$, $u_j = \frac{1}{\sqrt{2\pi}} e^{ijx}$, $j = 0, \pm 1, \pm 2, \cdots$, then λ_j^2, u_j respectively denote eigenvalue and eigenvector of an operator $A = -\frac{d^2}{dx^2} + I$, and $\text{span}\{u_j\} \subset H_p^1(D)$ is dense in $H_p^1(D)$. Let $u \in X$, $v \in Y$, then u and v can be written in the form

$$u = \sum_{j=-\infty}^{\infty} \alpha_j(t) u_j, \quad v = \sum_{j=-\infty}^{\infty} \beta_j(t) u_j,$$

with

$$\|u\|_X = \Big(\int_I \sum_{j=-\infty}^{\infty} \lambda_j^2 |\alpha_j(t)|^2 dt\Big)^{\frac{1}{2}},$$

$$\|v\|_Y = \Big(\int_I \sum_{j=-\infty}^{\infty} (\lambda_j^{-2} \mid \beta_j'(t) \mid^2 + \lambda_j^2 \mid \beta_j(t) \mid^2) dt\Big)^{\frac{1}{2}},$$

and $B(u,v)$ can also be written as follows

$$B(u,v) = \int_I \Big(\sum_{j=-\infty}^{\infty} (-\alpha_j \overline{\beta}_j' + \lambda_j^2 \alpha_j \overline{\beta}_j)\Big) dt = \int_I \Big(\sum_{j=-\infty}^{\infty} \lambda_j \alpha_j (-\lambda_j^{-1} \overline{\beta}_j' + \lambda_j \overline{\beta}_j)\Big) dt.$$