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ANALYSIS OF MOVING MESH METHODS BASED ON GEOMETRICAL VARIABLES*[1)]

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

In this study we will consider moving mesh methods for solving one-dimensional time dependent PDEs. The solution and mesh are obtained simultaneously by solving a system of *differential-algebraic* equations. The differential equations involve the solution and the mesh, while the algebraic equations involve several geometrical variables such as θ (the tangent angle), U (the normal velocity of the solution curve) and T (tangent velocity). The equal-arclength principle is employed to give a close form for T. For viscous conservation laws, we prove rigorously that the proposed system of moving mesh equations is well-posed, in the sense that first order perturbations for the solution and mesh can be controlled by the initial perturbation. Several test problems are considered and numerical experiments for the moving mesh equations are performed. The numerical results suggest that the proposed system of moving mesh equations is appropriate for solving (stiff) time dependent PDEs.

Key words: Moving mesh methods, Partial differential equations, Adaptive grids.

1. Introduction

Many methods have been proposed for adapting the mesh to achieve spatial resolution in the solution of partial differential equations. In addition to the capability of concentrating sufficient points about regions of rapid variation of the solution, a satisfactory mesh equation should be simple, easy to program, and reasonably insensitive to the choice of its adjustable parameters. The earliest work on adaptive techniques, based on moving finite element method (MFEM) was done by Miller [14, 12]. The gradient-weighted moving finite element (GWMFE) method was introduced recently by Miller as a geometrically motivated improvement over his earlier moving finite element methods. In [4, 5], Carlson and Miller reported on the design of the GWMFE codes and their extensive numerical trials on a variety of difficult PDEs and PDE systems. The *equidistribution principle*, first introduced by de Boor [7] for solving boundary value problems for ordinary differential equations, involves selecting mesh points such that some measure of the solution error is equalized over each subinterval. It has turned out to be an excellent principle for formulating moving mesh equations. In fact, a number of moving mesh methods have been

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developed, and almost all are based at some point on an equidistribution principle, see, e.g., [1, 2, 9, 15, 17].

In this work, we present a new method for generating numerical grids. The main motivation of this research is from the fundamental work of Hou, Lowengrub and Shelley [10] in which a new formulation was proposed for computing the motion of fluid interfaces with surface tension. One of the key ideas in their paper involves using a geometrical frame of reference so that the tangent angle of the interface θ and its length L, rather than its x and y position are the dynamical variables. With the θ-L formulation, the corresponding numerical methods have no high-order time step stability constraints that are usually associated with surface tension. The equal-arclength principle of de Boor is also employed in [10]. This idea enables them to express a geometrical variable T (tangent velocity) entirely in terms of θ and L. The problems investigated in [10] are of periodic solutions and therefore the θ-L formulation is an appropriate setting. In fact, the θ-L approach is useful not only for problems with periodic solutions but also for problems with Neumann boundary conditions. However, for commonly used Dirichlet boundary conditions the θ-L formulation may not be well-posed due to the unspecified boundary conditions for θ. In this case, we propose to solve a system of parametrized differential equations for $x = x(\alpha, t)$ and $\tilde{u} = u(x(\alpha, t), t)$. A system of differential-algebraic equations (DAEs) will be obtained, which involve $x, \tilde{u}$ and some geometrical variables. This system, together with the given boundary conditions for x and $\tilde{u}$, will be solved numerically.

The paper is organized as follows. In §2, we introduce the differential-algebraic formulations based on geometrical variables. The well-posedness of the numerical approach will be briefly investigated in §3. Some detailed numerical procedures will be discussed in §4. Numerical experiments will be carried out in the final section.

2. The Formulation

We consider again a single time evolving PDE in 1-D

$$u_t = \mathcal{F}(x, u, t) \tag{2.1}$$

with appropriate boundary and initial conditions, where $\mathcal{F}$ is some nonlinear spatial differential operator. As described in [13], we convert equation (2.1) into the normal form

$$U = (x_t, u_t) \cdot \mathbf{n} = \frac{u_t}{\sqrt{1 + u_x^2}}. \tag{2.2}$$

Now u is allowed to be an evolving oriented 1-D *manifold* immersed in two dimensions. Implicit in this geometrical treatment is the assumption that a choice of the ratio between the horizontal and vertical scales has been made and fixed.

The motion of interface is reposed in terms of its tangent angle $\theta(\alpha, t)$ and its local arclength derivatives $\sigma(\alpha, t) = \sqrt{x_\alpha^2 + \tilde{u}_\alpha^2}$. Derivations have been given elsewhere (e.g. [10]), but for completeness it is included here. The tangent angle to the curve Γ, θ, is the angle between $\mathbf{s}$ and the x-axis. It satisfies

$$\mathbf{s}(\alpha, t) = \left(\frac{x_\alpha(x, t)}{\sigma(\alpha, t)}, \frac{\tilde{u}_\alpha(\alpha, t)}{\sigma(\alpha, t)} \right) = \big(\cos\theta(\alpha, t), \sin\theta(\alpha, t) \big). \tag{2.3}$$

The unit vector in the normal direction, $\mathbf{n}$, is perpendicular to $\mathbf{s}$ and satisfies

$$\mathbf{n}(\alpha, t) = \big(-\sin\theta(\alpha, t), \cos\theta(\alpha, t) \big). \tag{2.4}$$