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FINITE ELEMENT APPROXIMATION OF A NONLINEAR STEADY-STATE HEAT CONDUCTION PROBLEM[*1)]

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

We examine a nonlinear partial differential equation of elliptic type with the homogeneous Dirichlet boundary conditions. We prove comparison and maximum principles. For associated finite element approximations we introduce a discrete analogue of the maximum principle for linear elements, which is based on nonobtuse tetrahedral partitions.

Key words: Boundary value elliptic problems, Comparison principle, Maximum principle, Finite element method, Discrete maximum principle, Nonobtuse partitions.

1. Introduction

Ancient Chinese mathematicians have done many fundamental discoveries, even though many of them are now usually called by western names, e.g., the Pascal triangle, the Horner scheme, the Gaussian elimination, see [23]. In the modern era, Chinese mathematicians have also got many priorities. For instance, the first proof of convergence of the finite element method for a linear elliptic boundary value problem was done in the pioneering work [6] by K. Feng in 1965 (for the English translation, see [7]). In 1968, M. Zlámal [30] proved a rate of convergence of this method. These results were later generalized to nonlinear problems (see, e.g., [10, 11, 14, 18, 19]).

During the development of the finite element method it has been found out that the rate of convergence of the finite element method at some exceptional points exceeds the optimal global rate. This phenomenon is known as superconvergence. Nowadays there are six research monographs on this theme [1, 3, 21, 22, 27, 29]. Five of them were written by Chinese mathematicians. For superconvergence of the finite element method in the case of nonlinear problems, see, e.g. [2, 4, 27].

In [18], we present a survey of results concerning convergence of the finite element method for a nonlinear steady-state heat conduction problem. The aim of this paper is to introduce some further properties valid for this problem, namely the maximum principle and its discrete version.

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Maximum principles play an important role in the theory of differential equations and mathematical modelling of physical phenomena. If the maximum principle would not be satisfied for a model of heat conduction in a body Ω, then some pathological situations could arise. For instance, the heat could flow from colder to warmer parts of Ω, i.e., such a model would not have reasonable physical properties. We could also obtain negative concentrations, densities etc. That is why so much attention is paid to maximum principles. They are mostly derived for the classical solutions of boundary or initial-boundary value problems of the second order, see [9, 24, 26].

In this paper we examine the nonlinear elliptic problem

$$-\mathrm{div}(A(\cdot\,,u)\mathrm{grad}\, u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, $d \in \{1, 2, ...\}$, $f \in L^2(\Omega)$, and $A = (a_{ij})_{i,j=1}^d$ is a uniformly positive definite matrix. Precise assumptions on the matrix function A are given in Section 2. In [11], we introduce sufficient conditions for the existence and uniqueness of u and we also give a one-dimensional example of nonuniqueness. The existence, uniqueness or multiplicity of solutions of similar nonlinear elliptic problems are examined, e.g., in [5, 8, 9, 10, 28].

The necessity to solve problem (1.1)–(1.2) arises in several real-life situations, e.g., in steady-state heat conduction in nonlinear inhomogeneous anisotropic media (see [19]). The matrix function A of heat conductivities depends on the unknown function u which represents the temperature and f is the density of volume heat sources.

2. Weak Formulation

To state a weak formulation of problem (1.1)–(1.2), we assume that the entries of $A = A(\cdot\,,\cdot)$ are bounded measurable functions, i.e.,

$$\max_{x,\xi,i,j} |a_{ij}(x,\xi)| \le C, \tag{2.1}$$

where $x \in \Omega$, $\xi \in \mathbb{R}^1$ and $i, j \in \{1, ..., d\}$. The entries a_{ij} are supposed to be Lipschitz continuous with respect to the last variable, i.e., there exists $C_L > 0$ such that for all $\zeta, \xi \in \mathbb{R}^1$ and almost all $x \in \Omega$ we have

$$|a_{ij}(x,\zeta) - a_{ij}(x,\xi)| \le C_L|\zeta - \xi|, \quad i, j = 1, ..., d. \tag{2.2}$$

Further, let there exist $C_0 > 0$ such that for almost all $x \in \Omega$

$$C_0\eta^T\eta \le \eta^T A(x,\xi)\eta \quad \forall \xi \in \mathbb{R}^1 \quad \forall \eta \in \mathbb{R}^d. \tag{2.3}$$

Finally, let $H^1(\Omega)$ be the standard Sobolev space of functions whose generalized first derivatives are square integrable. Denote by

$$V = H_0^1(\Omega)$$