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MIXED LEGENDRE–HERMITE PSEUDOSPECTRAL METHOD FOR HEAT TRANSFER IN AN INFINITE PLATE [*1)]

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Abstract

A new mixed Legendre-Hermite interpolation is introduced. Some approximation results are established. Mixed Legendre-Hermite pseudospectral method is proposed for non-isotropic heat transfer in an infinite plate. Its convergence is proved. Numerical results show the efficiency of this approach.

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Key words: Non-isotropic heat transfer, Infinite plate, Mixed Legendre-Hermite pseudospectral method.

1. Introduction

Spectral methods have been successfully used for numerical simulations of various problems in science and engineering, such as the Fourier spectral method for periodic problems, and the Legendre and Chebyshev spectral methods for bounded rectangular domains, see [2, 3, 5, 7, 9, 10]. Some authors also studied the Hermite spectral method for the whole line and the Laguerre spectral method for the half line, see [4, 6, 8, 11, 12, 16, 18, 19, 21].

There are three kinds of Hermite polynomial approximations. If the exact solutions grow fast at the infinity, then we usually take the Hermite polynomials $H_n(z)$ as the base functions as in [11], which are mutually orthogonal associated with the weight function e^{-z^2}. But in many cases, such as nonlinear wave equations, the solutions decay to zero at the infinity, and possess certain conservations which play important role in theoretical analysis. Thereby it seems better to use the Hermite functions $e^{-\frac{z^2}{2}}H_n(z)$ as in [13], which form the $L^2(-\infty,\infty)$-orthogonal system. Accordingly, the numerical solutions also keep certain conservations as in continuous cases. Furthermore, for some problems, such as heat transfer process, the solutions usually decay exponentially at the infinity. In this case, we prefer to the generalized Hermite functions $e^{-z^2}H_n(z)$, which are mutually orthogonal with respect to the weight function e^{z^2}, see [8].

In this paper, we consider non-isotropic heat transfer process in an infinite plate. The simplest way is to confine our calculation to a sufficiently large subdomain with certain artificial boundary condition. However, it causes additional errors. The authors proposed a mixed Legendre-Hermite spectral method for solving this problem, see [15]. However, it is more convenient to use pseudospectral method in actual computation, since we only need to evaluate unknown functions at the interpolation nodes. Especially, it is much easier to deal with nonlinear heat transfer process.

The aim of this paper is to develop the mixed pseudospectral method for non-isotropic heat transfer in an infinite plate, by using the Legendre interpolation in a direction, and the Hermite

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interpolation in another direction. As we know, there have been sharp results on the Jacobi interpolation, see [14]. Thus it suffices to study the Hermite interpolation precisely. Like the Hermite polynomial approximations, there have been also two kinds of Hermite interpolations which have the same interpolation nodes, but different weights of numerical quadratures, corresponding to the base functions $H_n(z)$ and $e^{-\frac{z^2}{2}}H_n(z)$, respectively, see [13, 16]. Whereas the solutions of transfer process decay exponentially at the infinity. Thus we introduce a new Hermite interpolation corresponding to the base functions $e^{-z^2}H_n(z)$, which could fit the asymptotic behavior of such solutions properly. Then we propose a new mixed Legendre-Hermite pseudospectral method for non-isotropic heat transfer in an infinite plate. Numerical results demonstrate the high accuracy in the space of this approach. We also establish some basic results on this new mixed Legendre-Hermite interpolation, from which the convergence of proposed scheme follows. These results also play important roles in forming and analyzing other related spectral methods for an infinite strip.

The paper is organized as follow. In the next section, we introduce the new mixed Legendre-Hermite interpolation and establish some basic approximation results. Then we describe the mixed Legendre-Hermite pseudospectral method and its implementation, and present some numerical results in Section 3. We prove the convergence of proposed scheme in Section 4. The final section is for concluding remarks.

2. Mixed Legendre-Hermite Interpolation

In this section, we introduce the new mixed Legendre-Hermite interpolation.

2.1 Legendre-Gauss-Lobatto interpolation

We first recall the Legendre-Gauss-Lobatto interpolation. Let $I = \{x|\ |x| < 1\}$. For any integer $r \geq 0$, we define the Sobolev space $H^r(I)$ as usual, with the inner product $(u, w)_{r,I}$, the semi-norm $|u|_{r,I}$ and the norm $||u||_{r,I}$. In particular, $(u, w)_I = (u, w)_{0,I}$ and $||u||_I = ||u||_{0,I}$. Moreover, $H_0^1(I) = \{\ u|\ u \in H^1(I) \text{ and } u(1) = u(-1) = 0\}$.

Denote by $L_m(x)$ the standard Legendre polynomial of degree $m, m = 0, 1, \cdots$. They satisfy the recurrence relation

$$(2m+1)L_m(x) = \partial_x L_{m+1}(x) - \partial_x L_{m-1}(x), \quad m \geq 1, \tag{2.1}$$

and form the $L^2(I)$-orthogonal system, i.e.,

$$\int_I L_m(x)L_{m'}(x)dx = \frac{2}{2m+1}\delta_{m,m'} \tag{2.2}$$

For any $u \in L^2(I)$, we have that

$$u(x) = \sum_{m=0}^{\infty} \hat{u}_m L_m(x), \qquad \hat{u}_m = (m+\frac{1}{2})\int_I u(x)L_m(x)dx. \tag{2.3}$$

For any integer $M \geq 0$, $\mathcal{P}_M$ stands for the set of all polynomials of degree at most M. Furthermore, $\mathcal{P}_M^0 = \{v \mid v \in \mathcal{P}_M,\ v(1) = v(-1) = 0\}$.

The orthogonal projection P_M: $L^2(I) \to \mathcal{P}_M$ is defined by

$$(P_M u - u,\ \phi)_I = 0, \qquad \forall\ \phi \in \mathcal{P}_M. \tag{2.4}$$

For description of approximation results, we introduce the space $H_A^r(I)$ with integer $r \geq 0$, equipped with the following semi-norm and norm

$$|u|_{r,A,I} = ||(1-x^2)^{\frac{r}{2}}\partial_x^r u||_I, \qquad ||u||_{r,A,I} = (\sum_{k=0}^{r}|u|_{k,A,I}^2)^{\frac{1}{2}}.$$