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CLOSED SMOOTH SURFACE DEFINED FROM CUBIC TRIANGULAR SPLINES *1)

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Abstract

In order to construct closed surfaces with continuous unit normal, we introduce a new spline space on an arbitrary closed mesh of three-sided faces. Our approach generalizes an idea of Goodman and is based on the concept of 'Geometric continuity' for piecewise polynomial parametrizations. The functions in the spline space restricted to the faces are cubic triangular polynomials. A basis of the spline space is constructed of positive functions which sum to 1. It is also shown that the space is suitable for interpolating data at the midpoints of the faces.

Mathematics subject classification: 41A05, 41A25.
Key words: Closed triangular mesh, Triangular Bernstein polynomial, Smooth spline, Geometric continuity.

1. Introduction

In computer-aided geometric design it is often useful to use surfaces defined parametrically from box splines on a regular mesh. Those box splines usually used are tensor product B-splines on a square mesh. However, such a representation cannot give a simple closed surface. It therefore seems natural to attempt to define a "geometrically smooth" simple closed surface. Attempts to do this have used the notion of subdivision for box splines to construct surfaces by a process of recursive subdivision of the mesh, see for examples [1],[2] and [3]. [6],[7] and [9] pioneered the idea of geometrically continuous spline spaces. [8] gives local bases of G^2 continuous G-splines. They consider only quadrilateral meshes. The other more general constructions based on irregular meshes may be seen in [5] and [10]. It is generally agreed that [6]-[9] are a global method in that one needs to solve large linear, irregularly sparse systems to match data, while [5] and [10] are a local method in that the coefficients of the parametrization in Bernstein-Bézier form are generated by applying averaging masks to the input mesh, but before computing the coefficients, several earlier algorithms contribute the idea of mesh refinement to parametrizations. In this paper we consider a global method similar to [9]. We take a closed polyhedral mesh M of three-sided faces and consider the space $S(M)$ of all functions on it whose restrictions to each face are certain polynomials and which satisfy certain matching conditions across the edges. These matching conditions ensure that a parametrically defined surface in R^3 whose components lie in $S(M)$ is G^1, i.e., is continuous and has a continuous unit normal vector.

After giving the G^1 continuous conditions between two adjacent triangular Bèzier patches in section 2, we consider in section 3 a way defining spline space $S(M)$ and discuss the dimension

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of $S(M)$. In section 4 we study in detail the special case when M is a tetrahedron and the technique used here should extend to other meshes. We construct a basis for $S(M)$ of positive functions which sum to 1 and are thus useful for the design of surfaces. We also show that $S(M)$ can be used for interpolating data at the midpoints of the faces of M.

2. The G^1 Continuous Conditions between Two Adjacent Triangular Bèzier Patches

2.1. Triangular Patches

In the section, we will present the G^1 continuous conditions between two adjacent triangular Bézier patches. Given conditions are used as the matching conditions of spline space $S(M)$ in next section. As for the G^1 continuous conditions, the discussion can also be seen in [4,11].

Triangular polynomial patches can be expressed in a Bernstein-Bèzier form,

$$\varphi(u,v,w) = \sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} G_{i,j,k}\frac{n!}{i!j!k!}u^i v^j w^k,\ u+v+w=1, u,v,w\geq 0, \tag{2.1}$$

where coefficients $G_{i,j,k}\in R^3$. The parameters u,v,w are called barycentric coordinates of a triangle; φ can be viewed as a map of this triangle into R^3(see Fig.1). Again, we use a shorthand notation for the coefficients:

$$T_i = G_{1,i,n-i-1},\ i=0,\dots,n-1,\ S_i = G_{0,i,n-i},\ i=0,\dots,n.$$

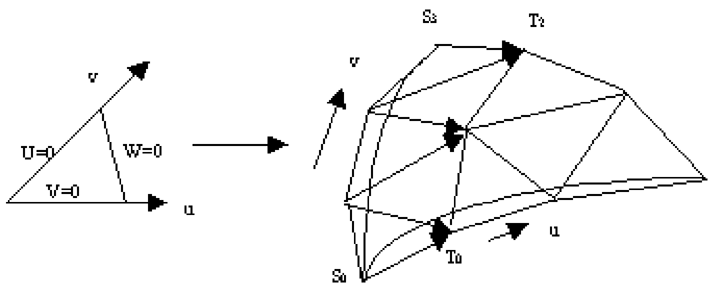

Fig.1. Triangular polynomial patch

The boundary $\Gamma(v) = \varphi(0,v,1-v)$ has the form

$$\Gamma(v) = \sum_0^n S_i B_i^n(v)$$

and hence its derivative is given by

$$[D\Gamma](v) = n\sum_{i=0}^{n-1}(S_{i+1}-S_i)B_i^{n-1}(v). \tag{2.2}$$

We shall consider a particular cross-boundary derivative, namely,

$$[D\varphi](v) = (1-v)(\varphi_u - \varphi_v) + v(\varphi_w - \varphi_v),$$

where $\varphi_u = \varphi_u(0,v,1-v), \varphi_v = \varphi_v(0,v,1-v), \varphi_w = \varphi_w(0,v,1-v)$. Expressed in terms of Bernstein polynomials,

$$[D\varphi](v) = n(1-v)\sum_{i=0}^{n-1}(T_i - S_i)B_i^{n-1}(v) + nv\sum_{i=0}^{n-1}(T_i - S_{i+1})B_i^{n-1}(v).$$