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CALCULATION OF NAMBU MECHANICS [*1)]

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper, a fundamental fact that Nambu mechanics is source free was proved. Based on this property, and via the idea of prolongation, finite dimensional Nambu system was prolonged to difference jet bundle. Structure preserving numerical methods of Nambu equations were established. Numerical experiments were presented at last to demonstrate advantages of the structure preserving schemes.

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1. Introduction

Nambu proposed an intriguing generalization of classical Hamiltonian mechanics [1], the idea was to extend the original Poisson bracket formulation of two function defined on R^2 to bracket of three functions h, f, g of $w = (x, y, z) \in R^3$, as

$$\{f, g, h\} = \frac{\partial(f, g, h)}{\partial(x, y, z)},$$

where $\frac{\partial(f,g,h)}{\partial(x,y,z)}$ is Jacobian. The equations of motion corresponding to f and g were written as

$$\dot{w} = \nabla f \times \nabla g. \tag{1}$$

Easy to see that for arbitrary function $h(x, y, z)$,

$$\frac{dh}{dt} = \{f, g, h\}. \tag{2}$$

Recent interest in this topic is due to Takhtajan [2] studied in particular the consistency requirement one should place on the generalization . Hietarinta [3] express the consistency conditions via Nambu tensor.

Briefly speaking, let $f_1, f_2, \cdots, f_n$ be n functions defined on

$$R^n = \{(x_1, x_2, \cdots, x_n) = w\},$$

Nambu bracket is defined as following

$$\{f_1, f_2, \cdots, f_n\} = \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)}. \tag{3}$$

It is easy to see that the bracket satisfys the following condition [2]:

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1 Skew symmetry

$$\{f_1, f_2, \cdots, f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, \cdots, f_{\sigma(n)}\},$$

where σ is a permutation of $1, \cdots, n$ and $\varepsilon(\sigma)$ is its parity.

2 The Leibnitz rule

$$\{f_1 g, f_2, \cdots, f_n\} = g\{f_1, f_2, \cdots, f_n\} + f_1\{g, f_2, \cdots, f_n\}.$$

1. Generalised Jacobi identity

$$\begin{aligned} & \{\{g_1, \cdots, g_{n-1}, f_1\}, f_2, \cdots, f_n\} + \{f_1, \{g_1, \cdots, g_{n-1}, f_2\}, \cdots, f_n\} \\ & +\{f_1, f_2, \cdots, \{g_1, \cdots, g_{n-1}, f_n\}\} \\ = \; & \{g_1, \cdots, g_{n-1}, \{f_1, f_2, \cdots, f_n\}\}. \end{aligned}$$

For given functions $f_1, f_2, \cdots, f_{n-1}$, we use f to denote the vector valued function $(f_1, f_2, \cdots, f_{n-1})^T$, the corresponding equations of motion are then

$$\begin{cases} \dot{x}_1 = \{f_1, f_2, \cdots, f_{n-1}, x_1\} = N(f_1, f_2, \cdots, f_{n-1})_1 = N(f)_1 \\ \dot{x}_2 = \{f_1, f_2, \cdots, f_{n-1}, x_2\} = N(f_1, f_2, \cdots, f_{n-1})_2 = N(f)_2 \\ \vdots \\ \dot{x}_n = \{f_1, f_2, \cdots, f_{n-1}, x_n\} = N(f_1, f_2, \cdots, f_{n-1})_n = N(f)_n \end{cases} \tag{4}$$

here we use $N(f_1, f_2, \cdots, f_{n-1}) == N(f)$ to denote Nambu vector fields generated by $f_1, f_2, \cdots, f_{n-1}$.

It will be proved in next section that the phase flow of (4) preserving $\omega^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. This make us construct structure preserving schemes: preserving ω^n, called volume preserving. Feng [4] proposed this idea for source free system, and established a systematic technique to design it via vector field splitting methods. Unfortunately this methods are unpractical, especially for large n, because it need first to split vector field to a sum of 2^n vector fields, the existence of such vector fields only be proved theoretically, then solve all the 2^n systems of differential equations numerically, the amount of calculation is as much as 2^n times of original one. There is until now no volume preserving schemes of source free systems have been found which is independent on the dimension of space and for arbitrary right term, only for special kind of systems, some first order methods have been designed out explicitly [4, 5]. By the technique of lift (4) to difference jet bundle, we established a systematical methods to design multi-step structure preserving schemes for Nambu mechanics. Numerical experiments have been made at last.

2. Phase Flow of Nambu System

In this section, we will mainly prove that phase flow of Nambu system preserves $n - form$

$$\omega^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

It is well known that the phase flow of source free system preserves ω^n, so we only need to prove (4) is source free, i.e.,

$$\frac{\partial}{\partial x_1}N(f)_1 + \frac{\partial}{\partial x_2}N(f)_2 + \cdots + \frac{\partial}{\partial x_n}N(f)_n = 0.$$

Theorem 1. *Nambu system* (4) *is a source free system*