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A UNIFIED A POSTERIORI ERROR ANALYSIS FOR DISCONTINUOUS GALERKIN APPROXIMATIONS OF REACTIVE TRANSPORT EQUATIONS *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

Four primal discontinuous Galerkin methods are applied to solve reactive transport problems, namely, Oden-Babuška-Baumann DG (OBB-DG), non-symmetric interior penalty Galerkin (NIPG), symmetric interior penalty Galerkin (SIPG), and incomplete interior penalty Galerkin (IIPG). A unified a posteriori residual-type error estimation is derived explicitly for these methods. From the computed solution and given data, explicit estimators can be computed efficiently and directly, which can be used as error indicators for adaptation. Unlike in the reference [10], we obtain the error estimators in $L^2(L^2)$ norm by using duality techniques instead of in $L^2(H^1)$ norm.

Mathematics subject classification: 65L10, 65L12.
Key words: A posteriori error estimates, Duality techniques, Discontinuous Galerkin methods.

1. Introduction

Numerical modeling of reactive transport problems in porous media is widely used in many fields, such as petroleum engineering, groundwater hydrology, environmental engineering, soil mechanics, earth sciences, chemical engineering and biomedical engineering. But, real simulations for simultaneous transport and chemical reaction present significant computational challenges [1, 2].

The discontinuous Galerkin (DG) method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems. Recently, the discontinuous Galerkin methods (DG) [3, 4, 5] have been popular for solving a wide variety of problems. DG has a lot of advantages over traditional finite element methods. Firstly, it is flexible which allows for general non-conforming meshes with variable degrees of approximation. secondly, it is locally mass conservative and the average of the trace of the fluxes along an element edge is continuous. Thirdly, it has less numerical diffusion and can deal with rough coefficient problems. Finally, it is easier for *h-p* adaptivity. DG applications for flow and transport problems in porous media

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have been studied in [6, 7].

A posteriori error estimators do not involve the knowledge of the exact unknown solution and are computable. At the same time, a posteriori error estimators are useful for adaptivity because they signify where refinement in spatial quantities or polynomial degree may be adaptively modified.

A posteriori error estimators for DG methods have mainly focused on steady-state equations of elliptic and hyperbolic type [8, 9]. And there are very few papers that deal with a posteriori error estimation for DG methods applied to transient problems. Explicitly, a posteriori error estimates in the $L^2(H^1)$ norm have been derived for four primal DG methods applied to reactive transport problems [10] without dual assumptions. Sun and Wheeler [11] derived an explicit $L^2(L^2)$ estimates for a symmetric discretization of the diffusion operator using a duality argument. In [12], $L^2(L^2)$ estimates of a non-symmetric interior penalty formulation and the related local discontinuous Galerkin formulation are explored. We remark that error indicators in the $L^2(L^2)$ norm are preferred over the indicators in $L^2(H^1)$ for problems concerning the concentration itself rather than the transport flux. In this paper, we will establish a unified a posteriori error estimation for four primal DG methods (i.e. OBB-DG, NIPG, SIPG, and IIPG) using duality techniques.

We consider a model reactive transport problem in a porous media

$$\phi\partial_t c + \nabla\cdot(\mathbf{u}c - \mathbf{D}\nabla c) = \phi f \quad in \quad \Omega, \quad t\in(0,T], \tag{1.1}$$

$$(\mathbf{u}c - \mathbf{D}\nabla c)\cdot\mathbf{n} = (\mathbf{u}g)\cdot\mathbf{n} \quad on \quad \Gamma_{in}, \quad t\in(0,T], \tag{1.2}$$

$$(-\mathbf{D}\nabla c)\cdot\mathbf{n} = 0 \quad on \quad \Gamma_{out}, \quad t\in(0,T], \tag{1.3}$$

$$c(x,0) = c_0(x) \quad in \quad \Omega. \tag{1.4}$$

where Ω is a polygonal and bounded domain in R^d ($d = 1, 2\ or\ 3$) with boundary $\partial\Omega = \Gamma_{in}\cup\Gamma_{out}$, $\Gamma_{in} = \{x\in\partial\Omega : \mathbf{u}\cdot\mathbf{n} < 0\}$ and $\Gamma_{out} = \{x\in\partial\Omega : \mathbf{u}\cdot\mathbf{n}\geq 0\}$ are the inflow boundary and the outflow boundary, $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$; $\mathbf{u}(x,t)$ represents the Darcy velocity and we assume that $\mathbf{u}$ is given and satisfies $\nabla\cdot\mathbf{u} = 0$; $c(x,t)$ is the concentration of some chemical component, $\phi(x)$ is the effective porosity of the medium and is bounded above and below by positive constants, $\mathbf{D}(x,\mathbf{u},t)$ denotes a diffusion or dispersion tensor and is uniformly positive definite, and $f(x,t)$ is a source term.

The paper is organized as follows. In section 2, we introduce the DG schemes. In section 3, a posteriori error estimators in $L^2(L^2)$ norm for the semi-discrete schemes are obtained using duality techniques explicitly. The numerical experiments are listed in section 4.

2. Discontinuous Galerkin Method

2.1. Notation

Let ε_h be a family of non-degenerate (or called regularity, which means that the element is convex and that there exists $\lambda > 0$ such that if h_j is the diameter of $E_j\in\varepsilon_h$, then each of the sub-triangles (for $d = 2$) or sub-tetrahedra (for $d = 3$) of element E_j contains a ball of radius λh_j in its interior), and possibly non-conforming finite element partitions of Ω composed of triangles or quadrilaterals if $d = 2$, or tetrahedra, prisms or hexahedra if $d = 3$.

Let Γ_h be the set of all interior edges (for 2 dimensional domain) or faces (for 3 dimensional domain) for ε_h. $\Gamma_{h,in}$ and $\Gamma_{h,out}$ denote the set of all edges or faces on Γ_{in} and Γ_{out} for ε_h, respectively. $\mathbf{n}_\gamma$ is the outward unit normal vector on each edge or face $\gamma\in\Gamma_h\cup\Gamma_{h,in}\cup\Gamma_{h,out}$.

The inner product in $(L^2(\Omega))^d$ or $L^2(\Omega)$ is indicated by $(\cdot,\cdot)_\Omega$ and the inner product in the boundary function space $L^2(\gamma)$ is indicated by $(\cdot,\cdot)_\gamma$.

For $s\geq 0$, we define

$$H^s(\varepsilon_h) = \{v\in L^2(\Omega) : v|_E\in H^s(E), E\in\varepsilon_h\}. \tag{2.1}$$