Journal of Computational Mathematics, Vol.24, No.1, 2006, 9–18.

SPLITTING EXTRAPOLATIONS FOR SOLVING BOUNDARY INTEGRAL EQUATIONS OF LINEAR ELASTICITY DIRICHLET PROBLEMS ON POLYGONS BY MECHANICAL QUADRATURE METHODS *1)

Jin Huang
(*University of electronic & science technology of China, Chengdu 610054; Mathematical College, Sichuan University, Chengdu 610064, China*)

Tao Lu
(*Mathematical College, Sichuan University, Chengdu 610064, China*)

Abstract

Taking h_m as the mesh width of a curved edge Γ_m $(m = 1, ..., d\ )$ of polygons and using quadrature rules for weakly singular integrals, this paper presents mechanical quadrature methods for solving BIES of the first kind of plane elasticity Dirichlet problems on curved polygons, which possess high accuracy $O(h_0^3)$ and low computing complexities. Since multivariate asymptotic expansions of approximate errors with power h_i^3 $(i = 1, 2, ..., d)$ are shown, by means of the splitting extrapolations high precision approximations and a posteriori estimate are obtained.

Mathematics subject classification: 65N38, 65R20, 75C35
Key words: Splitting extrapolation; Linear elasticity Dirichlet problem; Boundary integral equation of the first kind; Mechanical quadrature method

1. Introduction

Let $\Omega \subset R^2$ be curved polygons with the edges $\cup_{m=1}^{d}\Gamma_m = \Gamma$. Consider plane linear elasticity Dirichlet problems:

$$\begin{cases} Au \equiv \mu\Delta u + (\mu+\lambda)\mathrm{graddiv}u = 0, \text{ in } \Omega, \\ u = u^0, \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ is the displacement field, and μ and λ are Lame constants. By using the single layer potential theory, (1.1) can be converted into the following boundary integral equation system (BIES) of the first kind[2,3,17,18]

$$\int_\Gamma u_{ij}^*(y,x)p_j(x)ds_x = \alpha_{ij}(y)u_j^0(y) + \int_\Gamma p_{ij}^*(y,x)u_j^0(x)ds_x,\ i = 1,2,\ \forall y \in \Gamma, \tag{1.2}$$

where $y = (y_1, y_2)$, $r = |y - x| = [(y_1 - x_1)^2 + (y_2 - x_2)^2]^{1/2}$; $\alpha_{ij}(y)$ is a constant dependent on $y \in \Gamma$;

$$\begin{cases} u_{ij}^* = [-(3-4\nu)(\ln r)\delta_{ij} + r_{,.i}r_{,j}]/[8\pi\mu(1-\nu)], \\ p_{ij}^* = -[\partial r/\partial n((1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}) + (1-2\nu)(n_i r_{,j} - n_j r_{,i})]/[4\pi(1-\nu)r] \end{cases} \tag{1.3}$$

are Kelvin's fundamental solutions[2,3]; $\nu = \lambda/[2(\lambda+\mu)]$ is Poisson$'$s ratio; $n = (n_1, n_2)$ is the unit outward normal vector on Γ; $r_{,i} = \partial r/\partial x_i$ and repeated subscript means a summation from 1 to 2. Obviously, the equation (1.2) is a weakly singular boundary integral equation system

* Received January 14, 2004; Final revised June 6, 2005 .
1) Supported by Natioal Science Foundation of China (10171073).

of the first kind. If the traction $p = (p_1, p_2)^T$ is solved by (1.2), then the displacement vectors and stress tensor components can be calculated by

$$u_i(y) = \int_\Gamma u^*_{ij}(y,x)p_j(x)ds_x - \int_\Gamma p^*_{ij}(y,x)u^0_j(x)ds_x, \forall y \in \Omega, \tag{1.4}$$

$$\sigma_{ij}(y) = \int_\Gamma u^*_{ijk}(y,x)p_k(x)ds_x - \int_\Gamma p^*_{ijk}(y,x)u^0_k(x)ds_x, \forall y \in \Omega, \tag{1.5}$$

where

$$\begin{cases} u^*_{ijk} = [(1-2\nu)(r_{,j}\delta_{ki} + r_{,i}\delta_{kj} - r_{,k}\delta_{ij}) + 2r_{,i}r_{,j}r_{,k})]/[4\pi\mu(1-\nu)r]. \\ p^*_{ijk} = \mu/[2\pi(1-\nu)r^2]\{2\partial r/\partial n[(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}] \\ +2\nu(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}) + (1-2\nu)(2n_k r_{,i}r_{,j} + \delta_{ik}n_j + \delta_{jk}n_i) - (1-4\nu)\delta_{ij}n_k\}. \end{cases} \tag{1.6}$$

Unfortunately, the homogeneous equations corresponding to (1.2) might admit non-trivial solutions[7,23]. For simplicity, in the paper we assume

$$d(\Omega) = \max_{x,y\in\Gamma} |x-y| < 1, \tag{1.7}$$

which can ensure that the solution of (1.2) is unique (see Remark 1).

So far the numerical methods for solving (1.2) are Galerkin methods[7,22] and collocation methods [24] based on the projective theory, which have been applied to many engineering computations and application software. However there exist the following disadvantages: (1) Since the discrete matrix is full, the generating each element has to calculate an improper integral for the collocation method or a double improper integral for the Galerkin method, which implies that the work calculating discrete matrix is so large as greatly to exceed to solve the discrete equations. (2) The order of accuracy is lower, especially, for concave domain problems[22,24]. Obviously, using mechanical quadrature methods for solving (1.2) can save a lot of computations generating the discrete matrix. However, the convergent proof of the mechanical quadrature methods appears to be some difficulties without the projective theory as a mathematical tool. So far there are only a few papers to discuss the mechanical quadrature methods of the second-kind BIE[10]. In the paper, we propose a high accuracy mechanical quadrature method for solving the first-kind BIES of plane elasticity Dirichlet problems on curved polygons, which is based on the quadrature rules of the weakly singular periodic functions and the periodical transformations. Using the methods, we not only get the convergence rate of approximations, but also prove that the errors of the approximations possess the multivariate asymptotic expansions with power h_i^3 $(i = 1, 2, ..., d)$ given mesh widths. Thus as soon as some discrete equations with respect to some coarse mesh partitions are solved in parallel, the accuracy order of approximations will be improved by splitting extrapolation methods (SEM). Moreover, a posteriori asymptotic error estimate as adaptive algorithms is derived.

SEM[11,12,13] based on the multivariate asymptotic expansion of the error is a new extrapolation technique to solve large problems in parallel, which possesses a high order of accuracy, a high degree of parallelism and an almost optimal computational complexity. Since Lin and Lü published the first paper [12] in 1983, SEM has been applied to many multidimensional problems,e.g., the multidimensional numerical integrals[11,15], finite differential methods[11] and finite element methods[11]. Using Galerkin methods, Rüde and Zhou[19] gave SEM for solving the second kind BIE of Laplace's equation on polygonal domains. In the paper, SEM is first applied to solve the first kind BIES of the elasticity problems on curved polygons.

This paper is organized as follows: in Section 2 we derive a new integral equation system of the first kind with weakly singular kernels under the periodical transformations; in Section 3 the quadrature method and its convergent proof are given; in Section 4 the multivariate asymptotic expansions of the errors are derived, and SEM and a posteriori error estimate are got; in Section 5 some numerical examples are shown.

Remark 1. If (1.7) is satisfied, then it is easily verified that the matrix

$$\mathfrak{S} = \text{ı}8\pi\mu(1-\nu)\{-(3-4\nu)\ln r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r_{,1}r_{,1} & r_{,1}r_{,2} \\ r_{,1}r_{,2} & r_{,2}r_{,2} \end{bmatrix}\}.$$