Journal of Computational Mathematics, Vol.25, No.4, 2007, 421–429.

PARALLEL IMPLEMENTATIONS OF THE FAST SWEEPING METHOD [*1)]

Hongkai Zhao
(*Department of Mathematics, University of California, Irvine, CA 92697-3875, USA*
Email: zhao@math.uci.edu)

Abstract

The fast sweeping method is an efficient iterative method for hyperbolic problems. It combines Gauss-Seidel iterations with alternating sweeping orderings. In this paper several parallel implementations of the fast sweeping method are presented. These parallel algorithms are simple and efficient due to the causality of the underlying partial different equations. Numerical examples are used to verify our algorithms.

Mathematics subject classification: 65N06, 65N12, 65N55.
Key words: Hamilton-Jacobi equation, Eikonal equation, Characteristics, viscosity solution, Upwind difference, Courant-Friedrichs-Levy (CFL) condition, Gauss-Seidel iteration, Domain decomposition.

1. Introduction

The fast sweeping method is an iterative method that combines Gauss-Seidel iteration with alternating sweeping ordering. It was first introduced in [1] from the stochastic control point of view and then in [11] from partial differential equation point of view. For Eikonal type of equations it is fully analyzed and shown in [10] that the number of iteration is finite and is independent of mesh size. The fast sweeping method has been developed for unstructured meshes in [4] recently. The method has also been extended to more general Hamilton-Jacobi equations [2, 3, 5, 7] and high-order methods [9].

The convergence mechanism for fast sweeping method is quite different from the general framework of iterative methods. For most iterative methods, the convergence is due to some contraction property of the iteration. The fact that the fast sweeping method can converge in a finite number of iterations, e.g., for Eikonal equations which is a nonlinear hyperbolic boundary value problem, is because of the causality of the partial differential equation: information propagates along characteristics. With a systematic alternating ordering strategy all directions of characteristics can be divided into a finite number of groups and each group of directions is covered simultaneously by one of the orderings. Moreover, any characteristic can be covered by a finite number of orderings [10]. From the discrete point of view, the discretized system of nonlinear equations can be put into a triangular form if all nodes are ordered in an increasing order according to their values, i.e., if we interpret the viscosity solution to the Eikonal equation as the first arrival time for an expanding wave front later arrival time can be determined from earlier arrival time. A triangular system can be solved one equation by one equation in one sweep. Of course the problem is we do *not* know the solution so there is no such ordering a priori.

[*] Received December 20, 2006; final revised January 29, 2007; accepted March 3, 2007.
[1)] This work is partially supported by Sloan Foundation, NSF DMS0513073, ONR grant N00014-02-1-0090 and DARPA grant N00014-02-1-0603.

That is why systematic alternating orderings are needed to cover all possible causality blindly. In other words, with an appropriate upwind scheme that captures information propagation along characteristics correctly, we can characterize all nodes into a few groups. All nodes in each group have a similar dependence pattern on their neighbors. For example, using the monotone upwind difference scheme (2.2) on a rectangular grid in two dimensions for an Eikonal equation, all grid points can be divided into simply connected regions. In each region the value at a grid point depends on two of its neighbors in the following four ways: (1) left and down neighbors; (2) left and up neighbors; (3) right and down neighbors; (4) right and up neighbors. Of course there are possible degenerate cases where a grid point may only depend on one of its neighbors. In a degenerate case we can still classify it into one of the four cases above. For example, if a grid point only depends on its left neighbor we can either classify it into case (1) or (2). Using Gauss-Seidel iteration each connected region can be covered by one of the orderings simultaneously when the ordering is in the upwind direction of the dependence pattern. For example, the ordering $i = 1 : I, j = 1 : J$ is the upwind direction for the dependence pattern (1).

Since the fast sweeping method is an iterative method, a natural question is: can the fast sweeping method be parallelized? The answer is yes and simple. This is again due to the causality of the partial differential equation. We will present a few possible versions of parallel implementations of the fast sweeping method in Section 2 and show their efficiencies in Section 3 using numerical examples. Note that a parallel algorithm based on fast marching method (expanding front) was proposed in [8]. Although the above algorithm can achieve the optimal complexity $\mathcal{O}(n/p)$, where n is the total number of grid points and p is the number of processors, the algorithm works for a special discretization scheme. In addition a load balancing algorithm is needed at each stage. Our domain decomposition based parallel algorithms is very simple to implement and work for more general discretizations.

2. Parallel Algorithms For the Fast Sweeping Method

For simplicity we restrict our discussions to the computation for Eikonal equation in two dimensions on rectangular grids. Extensions to unstructured meshes and higher dimensions is straightforward. Assume we want to compute the positive viscosity solution of the following boundary value problem

$$\begin{array}{ll} |\nabla u(\boldsymbol{x})| = f(\boldsymbol{x}) & \boldsymbol{x} \in R^2, \\ u(\boldsymbol{x}) = 0 & \boldsymbol{x} \in \Gamma \subset R^2, \end{array} \tag{2.1}$$

where $f(\boldsymbol{x}) > 0$. Denote $\boldsymbol{x}_{i,j}$ to be a grid point in the computational domain Ω, h to be the grid size and $u^h_{i,j}$ to be the numerical solution at $\boldsymbol{x}_{i,j}$. First we give a brief description of the fast sweeping algorithm [10].

Discretization: At interior grid points Godunov type of upwind scheme is used to discretize the partial differential equation (2.1) which results in the following system of nonlinear equations

$$\begin{aligned} &[(u^h_{i,j} - u^h_{xmin})^+]^2 + [(u^h_{i,j} - u^h_{ymin})^+]^2 = f^2_{i,j} h^2 \\ &i = 2, \cdots, I-1, \quad j = 2, \cdots, J-1, \end{aligned} \tag{2.2}$$

where

$$u^h_{xmin} = \min(u^h_{i-1,j},\ u^h_{i+1,j}), \quad u^h_{ymin} = \min(u^h_{i,j-1}, u^h_{i,j+1})$$