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FULL DISCRETE TWO-LEVEL CORRECTION SCHEME FOR NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, a full discrete two-level scheme for the unsteady Navier-Stokes equations based on a time dependent projection approach is proposed. In the sense of the new projection and its related space splitting, non-linearity is treated only on the coarse level subspace at each time step by solving exactly the standard Galerkin equation while a linear equation has to be solved on the fine level subspace to get the final approximation at this time step. Thus, it is a two-level based correction scheme for the standard Galerkin approximation. Stability and error estimate for this scheme are investigated in the paper.

Mathematics subject classification: 65M55, 65M70.
Key words: Two-level method, Galerkin approximation, Correction, Navier-Stokes equation.

1. Introduction

We consider the two-dimensional Navier-Stokes equations

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \quad u(0) = u_0, \tag{1.1}$$

in certain divergence-free Hilbert space H, where u_0 is the initial velocity field, A the Stokes operator, B the projection of the non-linearity on H, $\nu > 0$ the kinetic viscosity and f the external force.

To get efficient numerical schemes, the two-level (two-grid) strategy has been widely studied. In particular, a class of two-level method in connection with the approximate inertial manifolds (AIMs) initialized by Foias, Manley and Temam [5] has been extensively studied in the past decades, which is usually called the nonlinear Galerkin method (NLG). Let ϕ_i be the ith eigenvector of the Stokes operator A corresponding to the associated eigenvalue λ_i. For given $m, M \in \mathbf{N}$ $(m < M)$, let P_m (P_M) denote the spectral projection from H onto the space spanned by the first m (M) eigenvectors. And we also set

$$Q_m = I - P_m \quad (Q_M = I - P_M).$$

The semi-discrete NLG reads: solve (1.1) up to a given time t_0 by a standard Galerkin method (SGM) in the fine-level subspace, and for $t > t_0$ find $v_m \in P_m H$ and $\hat{w}_m \in (P_M - P_m)H$ such that

$$\frac{dv_m}{dt} + \nu Av_m + P_m B(v_m + \hat{w}_m, v_m + \hat{w}_m) = P_m f, \tag{1.2}$$

$$\hat{w}_m = \Phi(v_m). \tag{1.3}$$

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Here Φ is the so-called AIM, a smooth mapping from P_mH onto $(P_M - P_m)H$ reflecting the approximate interactive relation between the lower and higher frequency components. For different choice of Φ we can get different NLG. For example, a frequently discussed AIM is expressed via the following generalized steady Stokes problem:

$$\nu A\hat{w}_m + (P_M - P_m)B(v_m, v_m) = (P_M - P_m)f. \tag{1.4}$$

This scheme is more efficient than SGM in P_MH. The convergence and error estimates for such NLG are obtained in the works of Marion and Temam [14, 15], Ammi and Marion [1], Marion and Xu [16] and Devulder et al. [4] in either finite element or spectral case. For example, in [4] they show that for $t > t_0$

$$\begin{aligned} &|u(t) - (v_m(t) + \hat{w}_m(t))|_{L^2} \le c(t)(L_m^3\lambda_{m+1}^{-\frac{3}{2}} + L_M\lambda_{M+1}^{-1}),\\ &|u(t) - (v_m(t) + \hat{w}_m(t))|_{H^1} \le c(t)(L_m^2\lambda_{m+1}^{-1} + L_M\lambda_{M+1}^{-\frac{1}{2}}), \end{aligned} \tag{1.5}$$

where $L_m = (1 + \ln\frac{\lambda_m}{\lambda_1})^{\frac{1}{2}}$. Lately, Garcia-Achilla et al. [6, 7] proposed a post-processing Galerkin scheme (PPG) based on the AIM defined by (1.4)

$$\frac{dv_m}{dt} + \nu Av_m + P_mB(v_m, v_m) = P_mf \quad \forall t \in [t_0, T], \tag{1.6}$$

$$\hat{w}_m(T) = \Phi(v_m(T)), \tag{1.7}$$

and obtained the similar error estimates. Since $\hat{w}_m$ is computed only once at $t = T$ and the lower and higher frequency components are fully dissociated, this is a very efficient scheme.

On the other hand, since the interaction of the lower and higher frequency components is reflected by a steady generalized Stokes equation, such schemes are only valid for $t > t_0$ when the time derivative of u possesses enough regularity. This is only acceptable for solutions slowly changed in higher frequency field. In fact, there have been few numerical experiments reported for such NLG and PPG to our knowledge, especially for highly oscillated solution in time field. Thus, to get reliable scheme for general cases, we should not neglect the self evolution of the higher frequency components. Another factor which affects the efficiency of the NLG or other two-level scheme is that they generally are coupled systems. When computing v_m we have to use $\hat{w}_m$ and vice versa. Such coupled systems are unavoidable if the space splitting are based on P_m and Q_m because such projections have nothing to do with the nonlinear system and the interaction of the different components has to be reflected by the coupled system they satisfied. Of course, PPG is an exception, in which the contribution of the higher frequency components $\hat{w}_m$ to itself and the lower frequency components v_m is neglected. This should be valid only for not very small viscosity case. Awaring of the reason of the generation of the coupled system, we alternate the way of thinking. If the decomposition of the lower and higher frequency components of the solution can reflect their interaction to some extent (for example, similar idea for steady state problems and certain scalar semi-linear evolutionary equations can be found in [11–13, 20]), it is reasonable to expect a two-level scheme in simpler form, at least in weakly coupled form, such that it is more efficient than usual two-level methods and can still preserve the order of convergence. These are the main motivation of this paper.

Based on the above considerations, we propose a full discrete two-level scheme for the Navier-Stokes equations: for given time step length $k > 0$, $u_M^n \in P_MH$ and $u_m^{n+1} \in P_mH$, find u_M^{n+1} in P_MH such that

$$(u_M^{n+1}, v) + k\nu(A^{\frac{1}{2}}u_M^{n+1}, A^{\frac{1}{2}}v) + k(B(u_m^{n+1}, u_M^{n+1}), v) = (u_M^n, v) + k(f^{n+1}, v), \tag{1.8}$$