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MULTI-PARAMETER TIKHONOV REGULARIZATION FOR LINEAR ILL-POSED OPERATOR EQUATIONS*

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Abstract

We consider solving linear ill-posed operator equations. Based on a multi-scale decomposition for the solution space, we propose a multi-parameter regularization for solving the equations. We establish weak and strong convergence theorems for the multi-parameter regularization solution. In particular, based on the eigenfunction decomposition, we develop a posteriori choice strategy for multi-parameters which gives a regularization solution with the optimal error bound. Several practical choices of multi-parameters are proposed. We also present numerical experiments to demonstrate the outperformance of the multi-parameter regularization over the single parameter regularization.

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Key words: Ill-posed problems, Tikhonov regularization, Multi-parameter regularization.

1. Introduction

The classical regularization method for solving ill-posed problems, which was proposed independently by Phillips [18] and Tikhonov [22], has been proved to be an excellent idea to overcome the difficulty caused by the ill-posedness, see [9, 12, 19, 23]. This regularization method turns an ill-posed problem to a well-posed problem which can be efficiently solved by standard numerical methods (cf. [1]). Such a method using a *single* parameter regularization is based on the hypothesis that noise *effect* to an ill-posed problem is uniformly distributed in all frequency bands of the solution. The single parameter regularization method adds a uniform penalty to every frequency band of the solution or the high-frequency band of the solution. The first case may result in solutions that are too smooth to preserve certain features of the original data. In the second case, the regularization solutions may be affected by low-frequency noise.

In practice, we observe different circumstances which lead us to consider *multi-parameter* regularization. Often, noise distributes differently in different parts of the physical domain. There is a case when noise distributes differently in different frequency bands. Sometimes noise has different effects to different frequency bands (scales) of the solution even though the noise is uniformly distributed. These circumstances suggest an introduction of multi-parameters to

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the regularization method. Multi-parameter regularization has been used in treating systems of linear equations in a few different contexts. A choice of multiple parameters was proposed in [2] by using the generalized L-curve method. A multi-parameter regularization algorithm for the solution of over-determined, ill-conditioned linear systems was proposed in [3], where numerical examples were presented to demonstrate that the proposed algorithm is stable and robust. In [8], the authors used a multi-parameter regularization method for atmospheric remote sensing. A multi-parameter regularization was used in [13] for solving a deconvolution problem in signal analysis when a wavelet transform was used to represent the system. The paper [14] proposed to use multi-parameter regularization methods based on biorthogonal wavelets and tight frame filter banks arising from the blurring kernel for treating the ill-posed problem related to high-resolution image reconstruction.

It is the main purpose of this paper to present convergence analysis for the multi-parameter regularization method for solving ill-posed operator equations when the solution space has a multi-scale decomposition (cf. [6, 10]). This paper will be based on the hypotheses that the function space and operators have a multi-scale structure and that noise has a different effect to a different frequency band of the solution due to the multi-scale structure of the solution, even though the noise is uniformly distributed. The proposed multi-parameter regularization will add different penalty parameters to different scales of the solution so that the ill-posedness is treated efficiently. At this point, we would like to point out that the multi-parameter regularization is more effective than single parameter regularization only if more information on the operator and the noise such as multi-scale decomposition is available.

The paper is organized into four sections. We describe in Section 2 the multi-parameter regularization method. Section 3 is devoted to the development of weak and strong convergence for the multi-parameter regularization solution. We also present an error estimate for the regularization solution and obtain a special result for regularization using the eigenfunction decomposition. In Section 4, we suggest a posteriori strategy for the choice of multi-parameters which gives a regularization solution with the optimal error bound when the eigenfunction decomposition is used. A numerical example is presented to illustrate the efficiency of this strategy. We also propose in Section 4 several practical strategies for the choice of the multiple parameters for the finite dimensional case, and present three numerical experiments in signal deconvolution and denoising using the multi-parameter regularization method. These experiments demonstrate the outperformance of the multi-parameter regularization over the single parameter regularization.

2. Multi-parameter Regularization Methods

We introduce in this section the multi-parameter regularization method for solving linear ill-posed operator equations based on a multi-scale decomposition of the solution space.

We first describe the linear ill-posed problem that we consider in this paper and recall the classical Tikhonov regularization method. Let $\mathbb{X}$ and $\mathbb{Y}$ be two Hilbert spaces. We will use $(\cdot,\cdot)$ for the inner product and $\|\cdot\|$ for the norm in both spaces without distinguishing them. Suppose that $\mathcal{K}:\mathbb{X}\to\mathbb{Y}$ is a linear compact operator. For a function $f\in\mathbb{Y}$, we consider the operator equation of the first kind

$$\mathcal{K}u = f. \tag{2.1}$$

We assume that the range $R(\mathcal{K})$ of operator $\mathcal{K}$ is of infinite dimension and thus, the solution of (2.1) does not continuously depend on the right-hand side f, that is, equation (2.1) is ill-posed