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A COMPARISON OF DIFFERENT CONTRACTION METHODS FOR MONOTONE VARIATIONAL INEQUALITIES*

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Abstract

It is interesting to compare the efficiency of two methods when their computational loads in each iteration are equal. In this paper, two classes of contraction methods for monotone variational inequalities are studied in a unified framework. The methods of both classes can be viewed as prediction-correction methods, which generate the same test vector in the prediction step and adopt the same step-size rule in the correction step. The only difference is that they use different search directions. The computational loads of each iteration of the different classes are equal. Our analysis explains theoretically why one class of the contraction methods usually outperforms the other class. It is demonstrated that many known methods belong to these two classes of methods. Finally, the presented numerical results demonstrate the validity of our analysis.

Mathematics subject classification: 65K10, 90C25, 90C30.
Key words: Monotone variational inequalities, Prediction-correction, Contraction methods.

1. Introduction

Let Ω be a nonempty closed convex subset of $\mathbb{R}^n$ and F be a continuous mapping from $\mathbb{R}^n$ into itself. A variational inequality problem, denoted by $\mathrm{VI}(\Omega, F)$, is to determine a vector $u^* \in \Omega$ such that

$$(u - u^*)^T F(u^*) \geq 0, \qquad \forall\, u \in \Omega. \tag{1.1}$$

$\mathrm{VI}(\Omega, F)$ problem includes nonlinear complementarity problem (when $\Omega = \mathbb{R}^n_+$) and system of nonlinear equations (when $\Omega = \mathbb{R}^n$) as its special cases and thus it has many applications [3,5]. The mapping F is said to be uniformly strong monotone (resp. monotone) on Ω if

$$(u - v)^T (F(u) - F(v)) \geq \mu \|u - v\|^2, \qquad \forall\, u, v \in \Omega,$$

where $\mu > 0$ (resp. $\mu = 0$) is a constant, F is Lipschitz continuous on Ω in the sense that there is a constant $L > 0$ such that

$$\|F(u) - F(v)\| \leq L \|u - v\|, \qquad \forall\, u, v \in \Omega.$$

Throughout this paper we assume that the operator F is monotone and Lipschitz continuous on Ω, and the solution set of $\mathrm{VI}(\Omega, F)$, denoted by Ω^*, is nonempty.

In the literature, there are different types of methods for monotone $\mathrm{VI}(\Omega, F)$ such as projection-contraction methods, continuous methods and cutting plane methods. Among these methods, the projection-contraction type of methods have attracted much attention for their simplicity. Let $P_\Omega(v)$ denote the projection of v onto Ω and u^k be the given current iterate. The

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simplest projection method is the Goldstein-Levitin-Polyak approach [4, 11] which iteratively updates u^{k+1} according to the formula

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)]. \tag{1.2}$$

This method produces a convergent sequence for uniformly strong monotone $\mathrm{VI}(\Omega, F)$ when $0 < \beta_L \le \beta_k \le \beta_U < 2\mu/L^2$. The basic projection method (1.2) is called an explicit method because all the terms in its right hand side are known. There are also implicit approaches (whose right hand side includes the unknown vector) such as the Douglas-Rachford operator splitting method [2, 12] which determines u^{k+1} by the recursion form

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)] + (F(u^k) - F(u^{k+1})) \tag{1.3}$$

and the proximal point algorithm [13] which generates u^{k+1} by

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^{k+1})]. \tag{1.4}$$

These implicit methods produce convergent sequences for monotone $\mathrm{VI}(\Omega, F)$ when $0 < \beta_L \le \beta_k \le \beta_U < +\infty$. The sequence $\{u^k\}$ generated by (1.4) satisfies

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - \|u^k - u^{k+1}\|^2, \quad \forall u^* \in \Omega^*.$$

The above inequality means that the new iterate u^{k+1} is closer to the solution set than the current point u^k. According to [1], the proximal point algorithm belongs to the class of Fejér contraction methods under Euclidean norm, or simply, contraction methods.

The main disadvantage of the implicit methods is that a subproblem should be solved in each iteration. Setting the u^{k+1} in (1.3) and (1.4) by u^k, we get the form (1.2), and the explicit method is convergent only for uniformly strong monotone (or co-coercive) $\mathrm{VI}(\Omega, F)$ when the parameter β_k is rigorously chosen. Instead of directly taking the left hand side of (1.2) as the new iterate, we set

$$\tilde{u}^k = P_\Omega[u^k - \beta_k F(u^k)] \tag{1.5}$$

as a predictor, the new iterate u^{k+1} (or called as corrector) will be generated by moving u^k in directions designed based on u^k and $\tilde{u}^k$. Such methods can be viewed as prediction-correction methods [9].

There are a number of contraction methods in the literature which belong to the prediction-correction methods. The purpose of this paper is to analyze the efficiency of the different methods whose computational loads in each iteration are equal. The paper is organized as follows. In section 2, we summarize preliminaries and define some basic concepts which will be used in this paper. Section 3 presents two criterions of the framework of the projection-contraction methods. In section 4, we analyze these two classes of methods theoretically and show that the iterates generated by the second class methods usually get more progress than those in the first class. Then, in section 5 we give linear and nonlinear applications with numerical experiments. As predicted by the analysis, the numerical results show the superiority of a class of methods clearly. Finally we give some conclusion remarks in section 6.

2. Preliminaries

Let G be an $n \times n$ positive definite matrix. The projection under G-norm is denoted by $P_{\Omega,G}(\cdot)$, *i.e.*,

$$P_{\Omega,G}(v) = \operatorname{argmin}\{\|v - u\|_G \mid u \in \Omega\}.$$