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IMinpert: An Incomplete Minimum Perturbation Algorithm for Large Unsymmetric Linear Systems

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Abstract

This paper gives the truncated version of the Minpert method: the incomplete minimum perturbation algorithm (IMinpert). It is based on an *incomplete orthogonalization* of the Krylov vectors in question, and gives a quasi-minimum backward error solution over the Krylov subspace. In order to make the practical implementation of IMinpert easy and convenient, we give another approximate version of the IMinpert method: A-IMinpert. Theoretical properties of the latter algorithm are discussed. Numerical experiments are reported to show the proposed method is effective in practice and is competitive with the Minpert algorithm.

Keywords: Minpert; nonsymmetric linear systems; backward error; iterative methods; Krylov subspace methods.
Mathematics subject classification: 65F10, 65F15, 65F50

1. Introduction

In many scientific and engineering computations, we want to solve the linear system of equations

$$Ax = b, \tag{1.1}$$

where A is an $n \times n$ real nonsymmetric matrix and b is an n-vector. A major class of methods for solving (1.1) is Krylov subspace type methods, and in these kinds of methods we usually use the residual error as a stopping condition. However, when A is ill conditioned, small residuals do not imply accurate approximate solutions. In order to overcome the disadvantage of using the residual error as a stopping criteria, Kasenally [1] proposed GMBACK algorithm, which computes an approximate solution restricted to an affine space while minimizing the backward perturbation norm of A:

$$\min_{x_m \in x_0 + \mathcal{K}_m(A, r_0)} \|\triangle_A\|_F \text{ subject to } (A - \triangle_A)x_m = b. \tag{1.2}$$

In the present setting, x_0 is an initial solution estimate and $r_0 := b - Ax_0$; x_m is the approximate solution to (1.1) of the form $x_m = x_0 + t_m$, where t_m belongs to the Krylov

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subspace; $\triangle_A$ and $\triangle_b$ are aggregated into a matrix $[\triangle_A, \triangle_b]$ which are known as the joint backward perturbation. Throughout this paper we shall make use of the Frobenious norm. Later, Kasenally and Simoncini, Cao Zhihao generalized the GMBACK method. They posed the Minpert algorithm [2,4], where the following minimization problem was solved:

$$\min_{x_m \in x_0 + \mathscr{K}_m(A, r_0)} \|[\triangle_A, \triangle_b]\|_F \text{ subject to } (A - \triangle_A)x_m = (b + \triangle_b). \tag{1.3}$$

We note that both GMBACK and Minpert algorithm employ the *Arnoldi process* to compute a matrix $V_m = [v_1, v_2, \ldots, v_m]$ whose columns form an orthogonal basis for $\mathscr{K}_m(A, r_0)$ [3], which means that both the methods use long recurrences. So work and storage per step grow drastically as the number of steps increases, and the methods thus become impractical for large steps. A popular technique is to resort to truncated strategies which only use a few rather than all the previously computed vectors in recurrences to get next vectors and thus be significantly less expensive than their non-truncated versions at each start. In this paper we will employ the *incomplete orthogonalization* [6,8,9] to compute V_m, and then get a truncated versions of Minpert: IMinpert. The only difference between Minpert algorithm and IMinpert algorithm is that the basis vectors $\{v_i\}_1^m$ are generated from different processes. We found from many numerical examples that the new method usually can get comparable results with the Minpert algorithm and when the Minpert method has better performances than restarted GMRES [2,4], the IMinpert also can present some advantages over restarted GMRES method.

The outline of this paper is as follows. Section 2 gives a new algorithm: an incomplete minimum perturbation algorithm. Implementation issues and another algorithm are introduced in Section 3. Section 4 gives some numerical experiments. Finally, the conclusions are provided in Section 5.

2. An incomplete minimum perturbation algorithm

2.1. Analysis of all joint backward perturbations

The following proposition parameterizes all perturbations for some approximation solution.

Proposition 2.1. *Suppose that m steps of the* incomplete orthogonalization process *have been taken. By construction the* incomplete orthogonalization process *yields an upper Hessenberg matrix* $H_m \in R^{(m+1)\times m}$ *which satisfies*

$$AV_m = V_{m+1}H_m. \tag{2.1}$$

Thus, by (1.3)*, any approximate solution may be written as*

$$x_m = x_0 + V_m y_m \textit{ for some } y_m \in R^m. \tag{2.2}$$

Let $\beta_m = \|r_0\|_2$. *The set of all joint backward perturbations* $S = [\triangle_A, \triangle_b]$ *such that* $(A - \triangle_A)x_m = (b + \triangle_b)$ *may be parameterized by R as*

$$S = \{V_{m+1}(H_m y_m - \beta e_1)\|[x_m^T, 1]\|_2^{-2}[x_m^T, 1] + R(I - ww^T) : R \in R^{n\times(n+1)}\}, \tag{2.3}$$